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CONTINUUM MODELS IN PROBLEMS OF THE HYPERSONIC FLOW OF A RAREFIED GAS AROUND BLUNT BODIES†

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All possible continuum (hydrodynamic) models in the case of two-dimensional problems of supersonic and hypersonic flows around blunt bodies in the two-layer model (a viscous shock layer and shock-wave structure) over the whole range of Reynolds numbers, Re, from low values (free molecular and transitional flow conditions) up to high values (flow conditions with a thin leading shock wave, a boundary layer and an external inviscid flow in the shock layer) are obtained from the Navier–Stokes equations using an asymptotic analysis. In the case of low Reynolds numbers, the shock layer is considered but the structure of the shock wave is ignored. Together with the well-known models (a boundary layer, a viscous shock layer, a thin viscous shock layer, parabolized Navier–Stokes equations (the single-layer model) for high, moderate and low Re numbers, respectively), a new hydrodynamic model, which follows from the Navier–Stokes equations and reduces to the solution of the simplified ("local") Stokes equations in a shock layer with vanishing inertial and pressure forces and boundary conditions on the unspecified free boundary (the shock wave) is found at Reynolds numbers, and a density ratio, k, up to and immediately after the leading shock wave, which tend to zero subject to the condition that $(k/\mathrm{Re})^{1/2} \to 0$. Unlike in all the models which have been mentioned above, the solution of the problem of the flow around a body in this model gives the free molecular limit for the coefficients of friction, heat transfer and pressure. In particular, the Newtonian limit for the drag is thereby rigorously obtained from the Navier–Stokes equations. At the same time, the Knudsen number, which is governed by the thickness of the shock layer, which vanishes in this model, tends to zero, that is, the conditions for a continuum treatment are satisfied. The structure of the shock wave can be determined both using continuum as well as kinetic models after obtaining the solution in the viscous shock layer for the weak physicochemical processes in the shock wave structure itself. Otherwise, the problem of the shock wave structure and the equations of the viscous shock layer must be jointly solved. The equations for all the continuum models are written in Dorodnitsyn–Lees boundary layer variables, which enables one, prior to solving the problem, to obtain an approximate estimate of second-order effects in boundary-layer theory as a function of Re and the parameter k and to represent all the aerodynamic and thermal characteristics in the form of a single dependence on Re over the whole range of its variation from zero to infinity.

An efficient numerical method of global iterations, previously developed for solving viscous shock-layer equations, can be used to solve problems of supersonic and hypersonic flows around the windward side of blunt bodies using a single hydrodynamic model of a viscous shock layer for all Re numbers, subject to the condition that the limit $(k/\mathrm{Re})^{1/2} \to 0$ is satisfied in the case of small Re numbers. An aerodynamic and thermal calculation using different hydrodynamic models, corresponding to different ranges of variation Re (different types of flow) can thereby, in fact, be replaced by a single calculation using one model for the whole of the trajectory for the descent (entry) of space vehicles and natural cosmic bodies (meteoroids) into the atmosphere.

The development of aerodynamic and thermal calculations on problems involving supersonic and hypersonic flows around blunt bodies in a trajectory for re-entry into the atmosphere is of great significance not only for improving existing space vehicles but, also, in the design of future, more economical space vehicles which manoeuvre themselves using aerodynamic forces (without propulsion) in the upper layers of the atmosphere [1, 2]. This is also important for predicting aerodynamics of flight, aerodynamic heating, ablation, the glow and the thermochemical and thermomechanical destruction of the heat protection of space probes [3], as well as for the quantitative prediction of the complex nature of the interaction between meteors and the Earth's atmosphere and that of planets during entry at super-orbital velocities [4].

At the present time, such calculations involve solving the corresponding hydrodynamic and kinetic equations, which is possible and reliable over a certain range of variation of Re numbers. On the whole, existing models, including kinetic models, encompass the whole range of flight Re numbers from low values at high altitudes up to high values at low altitudes and describe the whole of the perturbed domain of flow around the body.

Within the framework of continuum models for the flow around a body (low Knudsen numbers Kn), different hydrodynamic models have been used and are still used depending on the range of variation of the Re number and the degree of the gas compression in the shock layer (under conditions of hypersonic stabilization the Mach number M_∞ drops out from the system of similarity criteria) [5] and its indirect effect is associated with the occurrence of various physicochemical processes in the shock layer.

The system of Navier–Stokes equations is the most common kinetically and thermodynamically based continuum model. The solution of supersonic and hypersonic flow problems using this model is a laborious procedure, particularly in the case of long bodies and high Re numbers. The method of matched asymptotic expansions [6]

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was one of the first approximate approaches to solving the problem of the flow around a body at moderate and low Reynolds numbers. It is based on an extension of the Prandtl asymptotic scheme (a boundary layer plus an inviscid flow), which holds at high Reynolds numbers, to the domain of moderate and low Reynolds numbers when second-order effects in boundary-layer theory, which are of the order of $\mathrm{Re}^{-1/2}$, manifest themselves. These effects involve additional computations in the Prandtl scheme (first-order boundary-layer theory or the classical Prandtl scheme only takes account of terms of the order of unity when $\mathrm{Re} \to \infty$) of the longitudinal and transverse curvatures of the body around which the flow occurs, the displacement thickness, the gradients of the entropy and the enthalpy in the external boundary, the Navier–Stokes equations and the non-continuum effects of slip and of a temperature jump on the body surface [6]. Taking account of these effects using asymptotic expansions of the solutions of the Navier–Stokes equations in inverse powers of the square roots of the Re number leads to difficulties in obtaining uniformly valid solutions in the case of flows around long bodies in domains with a reduced pressure downstream, where a strong vortex interaction between the viscous and inviscid flow domains occurs [7]. As in any asymptotic approach, it is difficult to give an *a priori* estimate of the accuracy of the resulting solution for the fixed value of the parameter $\mathrm{Re}^{-1/2}$ in terms of which the solution is expanded.

A second approach is based on solving the viscous shock-layer (VSL) equations, that constitute a composite system of equations which follows from the Navier–Stokes equations and retains all second-order terms of boundary-layer theory in the viscous and inviscid flow domains [8–10]. In this model, the terms $O(1)$ and $O(\mathrm{Re}^{-1/2})$ are taken into account and only terms $O(\mathrm{Re}^{-1/2})$, which are responsible for molecular transfer of mass, momentum and energy along the body, are neglected. The system of VSL equations describes the propagation of perturbations upstream in subsonic flow domains and is of the elliptic type there. A numerical method of solution, which is extremely economical in its memory requirements and CPU time has been proposed in [11, 12] for solving the VSL equations. This is based on performing global iterations, which enable one to reduce the amount of CPU time required in the case of two-dimensional problems by a factor of approximately ten compared with step-by-step in time methods. The VSL model includes classical boundary-layer theory at high Reynolds numbers, but is restricted in its use when the Reynolds number tends to zero.

If, in the VSL equations, the ratio of the densities k up to and immediately after the leading shock wave tends to zero, then the pressure gradient normal to the body surface will be solely determined by centrifugal forces, and the effects of the longitudinal and transverse curvatures disappear. In this case, a simplified thin viscous shock-layer (TVSL) model is obtained which is of parabolic type and can be efficiently solved using the marching methods developed for solving the classical boundary-layer equations [3, 13, 14]. However, this model, in spite of its extensive use [3, 5], has a restricted region of application. In the problem of the flow around blunted cones, the TVSL model is only applicable in practice in the case of large semi-aperture angles of the cone ($\geqslant 40°$). A Newtonian separation with a zero pressure on the body occurs at smaller semi-aperture angles. Furthermore, specification of the shape of the shock wave equidistant from the body (in accordance with the asymptotic requirement of this model) leads, in the case of the flow around spherically blunted cones, to a "non-physical" discontinuity in the curvature of the shock wave and, as a consequence, to a discontinuity in the component of the pressure gradient tangential to the body [10, 15]. As a rule, at Re numbers corresponding to the appearance of a spread leading shock wave the full or simplified Navier–Stokes equations (see Section 8) with slip conditions and a temperature jump on the body are used [16]. This enables one to determine the heat fluxes and the friction force correctly up to Reynolds numbers $\mathrm{Re}_\infty = \rho_\infty V_\infty R_0/\mu_\infty \approx 100$ [17]. The solutions of the VSL equations and of the simplified and full Navier–Stokes equations when $\mathrm{Re} \to 0$ give infinite viscous friction and heat transfer coefficients. Kinetic equations are therefore used at the present time to determine the above coefficients and, in general, to determine the whole of the flow field when $\mathrm{Re} \to 0$ or the Monte-Carlo method is used [18, 19].

In this paper, the model of a continuum medium is extended to problems of hypersonic flow around blunt bodies at Reynolds numbers $\mathrm{Re} = \rho_\infty V_\infty R_0/\mu(T_0) \to 0$ (T_0 is the temperature of the adiabatically stagnant free stream). All of the above-mentioned models are obtained from the Navier–Stokes equations when $\mathrm{Re} \to \infty$. It is found that, in problems involving the hypersonic flow of a viscous gas around a body when $k \to 0$ and $\mathrm{Re} \to 0$, but subject to the condition that $(k/\mathrm{Re})^{1/2} \to 0$, the Navier–Stokes equations degenerate into simplified. Stokes equations which only contain second derivatives with respect to the coordinate normal to the body ("local" Stokes equations) with the usual no-slip conditions on the body and generalized Rankine–Hugoniot conditions on the side of the spread shock wave facing the body. The solution of this problem is written out in quadratures (Section 9) and gives the free-molecule limit for the pressure, drag and heat transfer coefficients. In this case, the ratio of the mean free path to the shock-layer thickness (the local Knudsen number, Kn) tends to zero.

This limiting value has a definite physical meaning and leads to local force and thermal laws for the interaction between a rarefied gas and the surface it flows about. In particular, Newton's formula for the pressure on a body is obtained in the above-mentioned limit. The boundary of the shock layer approaches the body in accordance with the law $(k/\mathrm{Re})^{1/2} \to 0$, and the whole of the domain of the perturbed flow around the body will therefore mainly be determined by the structure of the spread shock wave, which thickens as $\mathrm{Re} \to 0$. This structure can be determined after the flow in the shock layer has been found. The latter is found regardless of the shock-wave structure. This model of a vanishingly thin viscous shock layer (VTVSL) as well as the boundary-layer equations includes as a special case in the viscous shock-layer equations and provide the free-molecule limit under the above-mentioned conditions.

The viscous shock-layer (VSL), thin viscous shock-layer (TVSL) and vanishingly thin shock layer (VTVSL) models are based on boundary-layer equations as the basic model which contain all terms of the order of unity when $\mathrm{Re} \to \infty$. The VSL equations and all the simpler models which follow from them are therefore naturally written in the same Dorodnitsyn–Lees boundary-layer variables over the whole range of Reynolds numbers from zero to infinity, which encompasses all conditions for the flow around a body over the whole of the trajectory for its entry into the Earth's atmosphere (or into the atmosphere of a planet). These equations can be naturally solved using the same algorithm with simple "switching" (allowing k to approach zero) on approaching low Re numbers.

1. THE NAVIER–STOKES EQUATIONS IN A SYSTEM OF COORDINATES ATTACHED TO THE SURFACE OF THE BODY AROUND WHICH THE FLOW OCCURS

Assuming that the contour of a plane or an axially symmetric blunt body is sufficiently smooth, we shall consider the translationally steady-state flow of a gas around it with a velocity $\mathbf{V}_\infty$ which is directed along the body axis in an orthogonal curvilinear system of coordinates attached to the body surface. In this system of coordinates, the position of a point P in the flow is determined by its distance $y = PN$ along the normal to the contour, measured from the body surface, and by the length of the arc $x = ON$ along the contour, measured from its vertex O to the base of the normal N (Fig. 1).

We introduce the following notation: ρ_∞, V_∞ are the density and the velocity of the free stream and $\rho_\infty\rho$, $\rho_\infty V_\infty^2 p$ are the density and the pressure, respectively. Then, in the chosen system of coordinates, the Navier–Stokes equations for a homogeneous, viscous and heat-conducting gas will be [20]

$$\frac{\partial}{\partial x}\left(r^\nu \rho_\infty \rho v_1\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu \rho_\infty \rho v_2\right)=0 \tag{1.1}$$

$$\rho_\infty\rho\left(\frac{v_1}{H_1}\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_1}{\partial y}+\frac{v_1 v_2}{RH_1}\right)=$$
$$=-\frac{\rho_\infty V_\infty^2}{H_1}\frac{\partial p}{\partial x}+\frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu \tau_{xx}\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu \tau_{xy}\right)\right]+\frac{\tau_{xy}}{RH_1}-\frac{\nu\sin\alpha}{r}\tau_{\varphi\varphi} \tag{1.2}$$

$$\rho_\infty\rho\left(\frac{v_1}{H_1}\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}-\frac{v_1^2}{RH_1}\right)=$$
$$=-\rho_\infty V_\infty^2\frac{\partial p}{\partial y}+\frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu \tau_{xy}\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu \tau_{yy}\right)\right]-\frac{\tau_{xx}}{RH_1}-\frac{\nu\cos\alpha}{r}\tau_{\varphi\varphi} \tag{1.3}$$

$$\rho_\infty\rho\left(\frac{v_1}{H_1}\frac{\partial H}{\partial x}+v_2\frac{\partial H}{\partial y}\right)+\frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu J_x\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu J_y\right)\right]=0 \tag{1.4}$$

$$p=\rho R_A T / V_\infty^2 \tag{1.5}$$

In this system, Eq. (1.1) is the equation of continuity, (1.2) and (1.3) are the momentum equations in a projection onto the x and y axes, (1.4) is the energy equation in terms of the total enthalpy $H = h + (v_1^2 + v_2^2)/2$, and (1.5) is the equation of state which, for simplicity, is taken in the form for a perfect gas. By virtue of the symmetry of the problem

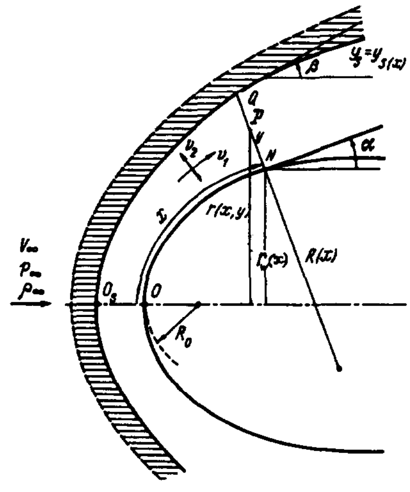

Fig. 1.

$$v_3 \equiv 0, \quad \partial/\partial\varphi \equiv 0 \tag{1.6}$$

Here x, y and φ are orthogonal curvilinear coordinates, v_1, v_2, v_3 are the physical components of the velocity vector of the gas in the orthonormalized basis of this system of coordinates, τ_{ij} $(i, j = x, y, \varphi)$ are the components of the viscous stress tensor, $H_1 = 1 + \varkappa(x)y \equiv 1 + R^{-1}(x)y$, $H_2 = 1$, $H_3 = r^\nu = [r_w(x) + y\cos\alpha(x)]^\nu$ are Lamé coefficients, $\varkappa(x)$, $R(x)$ are the curvature and the radius of curvature of the contour of the body, h is the thermodynamic enthalpy, R_A is the specific absolute gas constant, $R_A m$ is the absolute gas constant, m is the molecular mass of the gas, T is the temperature, $\nu = 0$ for plane flow and $\nu = 1$ for axially symmetric flows, $r_w(x)$ is the distance from a point on the contour around which the flow occurs to the axis of symmetry, $r(x, y)$ is the distance from a point in the flow to the axis of symmetry and $\alpha(x)$ is the angle between the tangent to the contour of the body around which the flow occurs and the axis of symmetry of the body (Fig. 1). In the case of a convex body, the functions $\alpha(x)$, $r_w(x)$, $R(x)$ and $r(x, y)$ are connected by the obvious geometric relations

$$\begin{gathered} r_w(x) = \int_0^x \sin\alpha(t)dt = \int_{\alpha(x)}^{\pi/2} R(t)\sin t\,dt \\ \frac{d\alpha}{dx} = \frac{1}{R(x)} = \varkappa(x), \quad \frac{\partial r}{\partial x} = H_1 \sin\alpha(x), \quad \frac{\partial r}{\partial y} = \cos\alpha(x) \end{gathered} \tag{1.7}$$

Certain relations between the components of the viscous stress tensor $\hat{\tau}$ and the components of the strain rate tensor $\hat{e}$, as well as an expression for the vector of the total energy flux density $\mathbf{J}$ are required to close the system of equations (1.1)–(1.5). We have [20]

$$\hat{\tau} = (\mu\zeta - 2\mu/3)\nabla\cdot\mathbf{v}\hat{G} + 2\mu\hat{e} \tag{1.8}$$

Here, μ, $\mu\zeta$ are the dynamic and bulk coefficients of viscosity and $\hat{G}$ is a metric tensor. The energy flux density vector can be expressed as follows:

$$\begin{gathered} \mathbf{J} = -\frac{\mu c_p}{\sigma}\nabla T + \hat{\tau}\cdot\mathbf{v} = -\frac{\mu}{\sigma}\left[\nabla H + \frac{\sigma}{\mu}(\hat{\tau}\cdot\mathbf{v}) - \nabla\frac{v^2}{2}\right], \quad \sigma = \frac{\mu c_p}{\lambda} \\ \hat{\tau}\cdot\mathbf{v} = (\tau_{xx}v_1 + \tau_{xy}v_2)\mathbf{e}_1 + (\tau_{xy}v_1 + \tau_{yy}v_2)\mathbf{e}_2, \quad v^2 = v_1^2 + v_2^2 \end{gathered} \tag{1.9}$$

where σ is the Prandtl number, λ is the thermal conductivity, c_p is the specific heat at constant pressure and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors along the x and y axes. When the dependences of the coefficients μ, $\mu\zeta$, λ and c_p on temperature are specified, the system of Navier–Stokes equations (1.1)–(1.5) is a closed system of five equations for determining the five functions: v_1, v_2, ρ, p, H (or T). Here, no account has been taken of the physicochemical processes which occur in the flow and accompany problems of hypersonic flow around a body [1, 9]. Taking account of these processes will not change the basic conclusions concerning the setting up of hydrodynamic models but it will, of course, change the quantitative results.

2. BOUNDARY CONDITIONS

For simplicity, we shall write out the boundary conditions on the body subject to the condition that the body is not thermochemically destroyed. Then, using the laws of conservation of mass and energy and the no-slip condition on the wall, we obtain

$$v_1(x, 0) = 0, \quad v_2(x, 0) = 0 \tag{2.1}$$

$$\frac{\mu}{\sigma}\frac{\partial H}{\partial y}(x, 0) = \varepsilon\sigma_B T_w^4(x, 0) \quad \text{or} \quad T(x, 0) = T_w(x) \tag{2.2}$$

Here, ε is the blackness of the body surface, σ_B is the Stefan–Boltzmann constant and $T_w(x)$ is the required temperature (the first condition of (2.2)) or the specified temperature (the second condition of (2.2)) of the body surface. The energy balance condition (2.2) is written assuming that the heat flux inside the body is negligibly small compared with the heat emission from the surface.

In the case of supersonic flow around a body, the velocity vector and the functions ρ, p and H (or T), consistent with the equation of state (1.5)

$$v_1[x, y_s(x)] = v_{1\infty}, \quad v_2[x, y_s(x)] = v_{2\infty}, \quad \rho[x, y_s(x)] = 1 \tag{2.3}$$
$$p[x, y_s(x)] = 1/(\gamma M_\infty^2), \quad H[x, y_s(x)] = H_\infty, \quad \gamma = c_p / c_v$$

must be specified in the free stream at infinity. Here, $y_s = y_s(x)$ is a conditional known boundary, located quite far ahead of the body on which the free stream parameters are specified and γ is the adiabatic exponent.

In the case of supersonic flow around a body at fairly high Reynolds numbers, when the thickness and structure of the leading shock wave can be neglected [21], it is convenient to replace the boundary conditions in the free stream by the corresponding conditions on the required shock wave. When Re $\to \infty$, these will be the usual Rankine–Hugoniot conditions at a strong discontinuity which, when the parameters are specified in the free stream, will give a one-parameter family of solutions for ρ, p, v_1, v_2 and H (or T) immediately outside the shock wave. This family of solutions depends on the angle of inclination of the shock wave $\beta(x)$. This angle and the standoff distance of the shock wave $y = y_s(x)$ are connected by the obvious geometrical relation (Fig. 1)

$$dy_s/dx = H_{1s} \operatorname{tg} \beta_s, \quad H_{1s} = 1 + y_s(x)/R(x), \quad \beta_s(x) = \beta - \alpha \tag{2.4}$$

where $\beta_s(x)$ is the angle of inclination of the shock wave to the x axis.

Equation (2.4) relates the two unknown quantities: $\beta_s(x)$ and $y = y_s(x)$ and, therefore, a single condition will not suffice when formulating the problem of supersonic flow around a body within the framework of the complete Navier–Stokes equations (1.1)–(1.5), which are of the seventh order with respect to the coordinate y, with four boundary conditions in the required shock wave and three conditions on the body (2.2).

In the literature, an additional condition $(\partial p/\partial y)(x, 0) = 0$ on the wall, which does not follow from the mechanical formulation of the problem, is often imposed or this formulation of the problem is closed at a difference level, which is done differently by different authors [10] and non-uniqueness of the solution is thereby produced. There is an alternative way of avoiding this artificial non-uniqueness: either to solve the problem without separating out the shock wave (it must be obtained when solving the problem), which is extremely difficult at high Reynolds numbers [22], or to reduce the order of the system of Navier–Stokes equations by one and formulate the boundary conditions on the required shock wave. The latter procedure is automatically implemented asymptotically in the two-layer model (the shock layer proper and the shock-wave structure) for the problem of flow around a body at both high and moderate Reynolds numbers [13, 14]. It is important to note that, in this case, just the out-of-order-of-magnitude terms, which are proportional to Re^{-1}, drop out of the Navier–Stokes equations. In order to take account of the conditions accompanying flow around a body at moderate Reynolds numbers, the conditions on the shock wave are written in the form of generalized Rankine–Hugoniot conditions, which take account of the viscosity and the heat conduction of the gas immediately behind the shock wave.

Dynamic compatibility conditions, which relate the characteristics of the gas motion up to and immediately after the shock wave while taking account of viscosity and heat conduction have been derived by Duhem and have been considered in detail by Kochin [23]. They have also been applied [24–27] to problems involving supersonic flow around a body. The compatibility conditions from the laws of conservation mass, momentum and energy, which are applied to a volume of fluid containing a discontinuity, neglecting the tangential components of the mass, momentum and energy flows within the structure of the shock wave as well as its thickness. In the case of the steady-state problem, when all the parameters are specified in the free stream, these conditions are

$$\rho_\infty v_{n\infty} = \rho_\infty \rho_s v_{ns}, \quad v_{n\infty} = -V_\infty \sin\beta \tag{2.5}$$

$$\rho_\infty v_{n\infty} \mathbf{v}_\infty + p_\infty \mathbf{n} = \rho_\infty \rho_s v_{ns} \mathbf{v}_s - \mathbf{p}_{ns} \tag{2.6}$$

$$\rho_\infty v_{n\infty} H_\infty = \rho_\infty \rho_s v_{ns} H_s + (\mathbf{J} \cdot \mathbf{n})_s \tag{2.7}$$

Quantities with the infinity subscript refer to conditions in the free stream while those with the subscript s refer to conditions on the required contour $y_s = y_s(x)$, $\mathbf{n}$ is the unit vector normal to the curve $y_s = y_s(x)$, directed towards the free stream and $\mathbf{p}_n$ is the stress vector in a small area with normal $\mathbf{n}$.

Together with (1.5), relations (2.5)–(2.7) in two-dimensional problems provide five equations for determining the five quantities: v_{1s}, v_{2s}, ρ_s, p_s, H_s (or T_s), which depend on the angle of inclination of the shock wave $\beta_s = \beta - \alpha$. The components of the velocity vector on the side of the shock-wave structure facing the body can be found from condition (2.5) and the vector condition (2.6) taken in a projection onto the tangent to the curve $y_s = y_s(x)$

$$v_{1s} = u_i - \frac{\cos\beta_s}{\rho_\infty V_\infty \sin\beta}\tau_{n\tau s} \tag{2.8}$$

$$v_{2s} = v_i - \frac{\sin\beta_s}{\rho_\infty V_\infty \sin\beta}\tau_{n\tau s} = v_{1s}\,\mathrm{tg}\,\beta_s - kV_\infty \frac{\sin\beta}{\cos\beta_s} \tag{2.9}$$

where

$$u_i = V_\infty \cos^2\beta_s[(1 + k\,\mathrm{tg}^2\beta_s)\cos\alpha - (1-k)\,\mathrm{tg}\,\beta_s \sin\alpha]$$

$$v_i = -V_\infty \cos^2\beta_s[(k + \mathrm{tg}^2\beta_s)\sin\alpha - (1-k)\,\mathrm{tg}\,\beta_s \sin\alpha] = u_i\,\mathrm{tg}\,\beta_s - kV_\infty \frac{\sin\beta}{\cos\beta_s} \tag{2.10}$$

$$\tau_{n\tau s} = \cos^2\beta_s[\tau_{xy}(1 - \mathrm{tg}^2\beta_s) + (\tau_{yy} - \tau_{xx})\,\mathrm{tg}\,\beta_s]_s$$

Here, u_i, v_i are the components of the velocity vector along the x and y axes for the shock wave in an ideal gas (when there is no viscosity), $\tau_{n\tau s} = p_{n\tau s}$ is the projection of the viscous stress vector $\hat{\tau}\cdot\mathbf{n}$ onto the tangent to the contour $y_s = y_s(x)$ and $k = \rho_\infty/\rho_\infty\rho_s = \rho_s^{-1}$ is the ratio of the density in the free stream to the density beyond the jump (on the required contour $y_s = y_s(x)$).

From the remaining equations (2.6) and (2.7) and the equation of state (1.5), we obtain the remaining quantities on the unknown contour

$$p_s = \frac{1}{\gamma M_\infty^2} + (1-k)\sin^2\beta + \frac{\tau_{nns}}{\rho_\infty V_\infty^2}$$

$$H_s = H_\infty + \frac{1}{\rho_\infty V_\infty \sin\beta}(\mathbf{J}\cdot\mathbf{n})_s \tag{2.11}$$

$$\frac{1}{\rho_s} \equiv k = \frac{R_A}{V_\infty^2}\frac{T_s}{p_s} = \frac{\gamma - 1}{2\gamma p_s}\frac{T_s}{T_0}$$

where

$$\tau_{nns} = \cos^2\beta_s(\tau_{yy} + \tau_{xx}\,\mathrm{tg}^2\beta_s - 2\tau_{xy}\,\mathrm{tg}\,\beta_s)_s$$

$$(\mathbf{J}\cdot\mathbf{n})_s = \cos\beta_s(J_{Hy} - J_{Hx}\,\mathrm{tg}\,\beta_s)_s, \quad T_0 = \frac{V_\infty^2}{2c_p} \tag{2.12}$$

T_0 is the temperature of the adiabatically stagnant free stream (minus the temperature of the free stream).

Relations (2.8), (2.9) and (2.11) become the Rankine–Hugoniot relations on a shock wave of zero thickness if the viscosity and thermal conductivity of the gas are neglected, that is, when $\mathrm{Re} \to \infty$. In this case, for a specified angle $\beta_s = \beta_s(x)$, the five relations (2.8), (2.9) and (2.11) are sufficient to determine the five parameters for the shock wave. When the gas has viscosity and thermal conductivity, this cannot be done prior to solving the problem.

Conditions (2.8), (2.9) and (2.11) are boundary conditions, which are sometimes called slip conditions on account of the discrepancy between the tangential component of the velocity and the total enthalpy across the shock wave and the corresponding parameters in an inviscid gas due to the effects of viscosity and thermal conductivity. The velocity component which is tangential to the shock wave ($u_{\tau s} \neq u_{\tau\infty}$) and the total enthalpy ($H_s \neq H_\infty$), which remain continuous across the jump in an ideal gas, suffer a discontinuity.

After solving the problem, it is usually necessary to find the pressure coefficient

$$C_p = \frac{\rho_\infty V_\infty^2 p_w - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\left(p_w - \frac{1}{\gamma M_\infty^2}\right) \quad (2.13)$$

the friction coefficient

$$C_f = \frac{\tau_{xyw}}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{2}{\rho_\infty V_\infty^2}\left(\mu \frac{\partial v_1}{\partial y}\right)_w \quad (2.14)$$

the convective heat transfer coefficient

$$C_H = \frac{q_w}{\rho_\infty V_\infty (H_\infty - h_w)} = \left(\frac{\mu}{\sigma}\frac{\partial H}{\partial y}\right)_w \frac{1}{\rho_\infty V_\infty (H_\infty - h_w)} \quad (2.15)$$

and the total drag coefficient of the plane contour ($\nu = 0$) (per unit length of the wing span perpendicular to the free stream velocity) or of the body of revolution ($\nu = 1$) with an arc length x

$$C_D = \frac{2F_z}{\rho_\infty V_\infty^2 \pi^\nu r_w^{\nu+1}(x)} = \frac{4}{\rho_\infty V_\infty^2 r_w^{\nu+1}} \int_0^x \left[\tau_{xy}\cos\alpha + \left(\rho_\infty V_\infty^2 p_w - p_\infty - \tau_{yy}\right)\sin\alpha\right]_w r_w^\nu dx =$$

$$= \frac{4}{\rho_\infty V_\infty^2 r_w^{\nu+1}} \int_0^x \left[\left(\mu \frac{\partial v_1}{\partial y}\right)_w \cos\alpha + \left(\rho_\infty V_\infty^2 p_w - p_\infty\right)\sin\alpha\right]_w r_w^\nu dx =$$

$$= \frac{2}{r_w^{\nu+1}} \int_0^x \left(C_f \cos\alpha + C_p \sin\alpha\right) r_w^\nu dx \quad (2.16)$$

where F_z is the total axial force acting on the windward part of the body with a length of the contour x.

3. THE NAVIER–STOKES EQUATIONS AND BOUNDARY CONDITIONS IN DORODNITSYN–LEES VARIABLES

In the analytic and numerical solution of problems in aerodynamics and heat exchange it is important that both the independent and the dependent variables are chosen in a rational manner. In boundary-layer theory, the use of Dorodnitsyn variables [28] in the Lees form [29] leads to a weaker dependence of the required functions on the coefficients of the equations than when the initial (physical) variables are used, a weak dependence (in the case of laminar flows) of the boundary-layer thickness on the longitudinal coordinate and possibilities for obtaining self-similar and quasi-self-similar solutions under certain conditions. Furthermore, it has been shown that, in Dorodnitsyn variables within the framework of the thin inviscid shock layer model, the shock-layer thickness, the velocity profiles and the longitudinal pressure gradient in the neighbourhood of the critical point are independent of the density profiles along the axis, the variability of which may be caused by the compressibility or other physicochemical processes in the shock layer [30]. In the above-mentioned variables a weak dependence of the standoff distance of the leading shock wave on certain parameters of the problem is established and, also, similarity laws [31].

Since all second-order effects in boundary-layer theory are corrections to the results for a classical boundary layer and are contained in the composite system of viscous shock-layer (VSL) equations (Section 6), which differs from the complete system of Navier–Stokes equations in the small terms O ($\mathrm{Re}^{-1} \sim \mathrm{Kn}$), it is natural to write the initial system of Navier–Stokes equations (1.1)–(1.5) and the simplified models which follow from this system in Dorodnitsyn–Lees variables. The idea is therefore developed in this paper that, if Dorodnitsyn–Lees variables are effective when solving problems in boundary-layer theory, they will also be effective when the solutions of problems of supersonic flow around a body are represented using more complex gas dynamic models and, in particular, the VSL model which also includes the complete Navier–Stokes equations.

So, we transform the Navier–Stokes equations (1.1)–(1.5) and the boundary conditions on the body and on the shock wave to the new independent variables

$$\xi = \xi(x), \quad \eta = \frac{1}{\Delta(x)} \int_0^y \rho \bar{r}^\nu dy, \quad \bar{r} = \frac{r}{r_w} = 1 + y\frac{\cos\alpha}{r_w} \tag{3.1}$$

Here, $\xi(x)$ and $\Delta(x)$ are, at present, arbitrary functions which are subsequently selected from considerations of the ease of writing down the equations and normalizing the physical thickness of the boundary layer or of the shock layer and, also, to make it easier to obtain estimates and numerical solutions. The inverse transform of (3.1) will be

$$x = x(\xi), \quad \bar{r}^{\nu+1} = 1 + (\nu+1)\Delta\frac{\cos\alpha}{r_w}\int_0^\eta \frac{1}{\rho}d\eta \tag{3.2}$$

Using Eq. (1.1), we now determine the stream function $(2\pi)^\nu\psi(x, y)$ from the system of equations

$$\partial\psi / \partial x = -H_1 r^\nu \rho_\infty \rho v_2, \quad \partial\psi / \partial y = r^\nu \rho_\infty \rho v_1 \tag{3.3}$$

and we shall seek $\psi(x, y)$ in the new variables in the form

$$\psi(x, y) = b(x) f(\xi, \eta), \quad b(x) = \rho_\infty u_* r_w^\nu \Delta \tag{3.4}$$

The function $f(\xi, \eta)$ is called the reduced stream function and u_* is the characteristic velocity which is chosen in each model in its own way. If a further characteristic velocity υ_* is introduced and one defines the dimensionless projections of the velocity

$$u = v_1/u_*, \qquad v = v_2/v_* \tag{3.5}$$

then Eqs (3.3) become

$$u = \partial f/\partial\eta, \qquad \rho(k_1 u + k_2 v) = -(\beta_0 f + x\xi'(x)\partial f/\partial\xi) \tag{3.6}$$

The coefficients k_1, k_2, β_0 are given by expressions (3.13), which are presented below.

We now introduce the dimensionless components of the viscous stress tensor $\tau_{ij}(\xi, \eta)$ $(i, j = \xi, \eta, \zeta)$ and the dimensionless projections of the energy flux vector $X(\xi, \eta)$ and $Y(\xi, \eta)$ using the formulae

$$\tau_{xx}(x, y) = \frac{\mu u_*}{xH_1}\tau_{\xi\xi}(\xi, \eta), \quad \tau_{yy}(x, y) = \frac{\mu u_*}{xH_1}\tau_{\eta\eta}(\xi, \eta)$$

$$\tau_{\varphi\varphi}(x, y) = \frac{\mu u_*}{xH_1}\tau_{\zeta\zeta}(\xi, \eta)$$

$$\tau_{xy}(x, y) = \frac{\mu u_* \rho \bar{r}^\nu}{\Delta}\tau_{\xi\eta}(\xi, \eta) = \frac{b u_*}{xH_1 r^\nu \operatorname{Re}^* \Delta^2}\tau_{\xi\eta}(\xi, \eta) \tag{3.7}$$

$$J_x(x, y) = -\frac{\mu}{xH_1}X(\xi, \eta)$$

$$J_y(x, y) = -\frac{\mu\rho\bar{r}^\nu}{\Delta\sigma}Y(\xi, \eta) = -\frac{b}{xH_1 r^\nu \operatorname{Re}^* \Delta^2\sigma}Y(\xi, \eta)$$

$$\operatorname{Re}^* = \frac{\rho_\infty u_*}{\mu\rho x H_1 \bar{r}^{2\nu}} \tag{3.8}$$

Then, when account is taken of Eqs (3.6), the remaining equations (1.2)–(1.4) in the variables (3.1) can be written in the form

$$\beta_1 u^2 + k_3 u\upsilon + x\xi'(x)u\frac{\partial u}{\partial \xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial u}{\partial \eta} =$$

$$= -\frac{V_\infty^2}{\rho u_*^2}\left(x\xi'(x)\frac{\partial p}{\partial \xi} + \rho k_1 \frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\mathrm{Re}^* \Delta^2}\tau_{\xi\eta}\right) + \frac{k_6}{\mathrm{Re}^* \Delta^2}\tau_{\xi\eta} + \varepsilon_1 \tau'_{\xi\xi} - \varepsilon_2 \tau_{\zeta\zeta} \tag{3.9}$$

$$\beta_2 u\upsilon - \frac{1}{k_4}u^2 + x\xi'(x)u\frac{\partial \upsilon}{\partial \xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial \upsilon}{\partial \eta} =$$

$$= -\frac{1}{k_5}\frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \eta}(\varepsilon_3 \tau_{\eta\eta}) + \varepsilon_3 \tau'_{\xi\eta} - \varepsilon_4 \tau_{\xi\xi} - \varepsilon_5 \tau_{\zeta\zeta} \tag{3.10}$$

$$x\xi'(x)u\frac{\partial H}{\partial \xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial H}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{1}{\mathrm{Re}^* \Delta^2 \sigma}Y\right) + \varepsilon_1 X' \tag{3.11}$$

Here

$$\tau_{\xi\xi} = 2(\beta_1 u + k_3 \upsilon + Du) + (\zeta - 2/3)\nabla' \cdot \mathbf{v}$$

$$\tau_{\eta\eta} = 2\rho k_2 \frac{\partial \upsilon}{\partial \eta} + (\zeta - 2/3)\nabla' \cdot \mathbf{v}$$

$$\tau_{\xi\xi} = 2\nu(n_1 u + k_7 \upsilon) + (\zeta - 2/3)\nabla' \cdot \mathbf{v}$$

$$\tau_{\xi\eta} = \frac{\partial u}{\partial \eta} - k_6 u + k_8(\beta_2 \upsilon + D\upsilon)$$

$$\tau'_{\xi\xi} = m_1 \tau_{\xi\xi} + \frac{1}{\mu}D(\mu \tau_{\xi\xi}) \tag{3.12}$$

$$\tau'_{\xi\eta} = m_2 \tau_{\xi\eta} + \mathrm{Re}^* \Delta^2 D\left(\frac{1}{\mathrm{Re}^* \Delta^2}\tau_{\xi\eta}\right)$$

$$\nabla' \cdot \mathbf{v} = \beta_1 u + k_3 \upsilon + \nu(n_1 u + k_7 \upsilon) + Du + \rho k_2 \partial \upsilon / \partial \eta$$

$$X = DH - D(\upsilon_1^2 + \upsilon_2^2)/2 + \sigma u_*^2(u\tau_{\xi\xi} + \rho k_2 \upsilon\, \tau_{\xi\eta})$$

$$X' = m_3 X + \frac{1}{\mu}D(\mu X)$$

$$Y = \frac{\partial H}{\partial \eta} - \frac{1}{2}\frac{\partial}{\partial \eta}(\upsilon_1^2 + \upsilon_2^2) + \sigma u_*^2\left(u\tau_{\xi\eta} + \frac{\upsilon_*^2}{u_*^2}\frac{\upsilon}{\rho k_2}\tau_{\eta\eta}\right)$$

$$D = x\xi'(x)\frac{\partial}{\partial \xi} + \rho k_1 \frac{\partial}{\partial \eta}$$

A number of dimensionless coefficients appear in the system of equations (3.6), (3.9)–(3.11) which are expressed both in terms of specified quantities and quantities which are unknown until the problem has been solved u_*, υ_*, R, r_w, α, ρ, Δ, y

$$\beta_0 = \frac{d\ln b}{d\ln x},\quad \beta_1 = \frac{d\ln u_*}{d\ln x},\quad \beta_2 = \frac{d\ln \upsilon_*}{d\ln x}$$

$$m_1 = \beta_1 + \frac{\nu}{r} x H_1 \sin\alpha + \frac{y}{RH_1}\frac{d\ln R}{d\ln x} - 1$$

$$m_2 = 2\beta_1 + \frac{d\ln\Delta}{d\ln x} + \frac{\nu}{r_w} x \sin\alpha + \frac{y}{RH_1}\frac{d\ln R}{d\ln x} - 1 \tag{3.13}$$

$$m_3 = m_1 - \beta_1, \quad n_1 = \frac{xH_1 \sin\alpha}{r_w}$$

$$k_1 = \frac{x}{\rho}\frac{\partial\eta}{\partial x} = -\frac{x\xi'(x)\bar{r}^{\nu}}{\Delta}\frac{\partial y}{\partial\xi}, \quad k_2 = \frac{xH_1\bar{r}^{\nu}v_*}{u_*\Delta}, \quad k_3 = \frac{xv_*}{Ru_*}$$

$$k_4 = \frac{Rv_*}{xu_*}, \quad k_5 = \frac{u_* v_* \Delta}{xH_1\bar{r}^{\nu}V_\infty^2}, \quad k_6 = \frac{\Delta}{\rho H_1 \bar{r}^{\nu} R}$$

$$k_7 = \frac{xH_1 v_* \cos\alpha}{ru_*}, \quad k_8 = \frac{v_*\Delta}{\rho u_* x H_1 \bar{r}^{\nu}}$$

$$\varepsilon_1 = \frac{\mu}{\rho_\infty \rho u_* x H_1}, \quad \varepsilon_2 = \frac{\mu \sin\alpha}{\rho_\infty \rho u_* r}, \quad \varepsilon_3 = \frac{\mu \bar{r}^{\nu}}{\rho_\infty v_* \Delta}$$

$$\varepsilon_4 = \frac{\mu}{\rho_\infty \rho v_* R H_1}, \quad \varepsilon_5 = \nu\frac{\mu\cos\alpha}{\rho_\infty \rho v_* r}$$

We now eliminate the derivative $\partial p/\partial\eta$ from Eq. (3.9) using Eq. (3.10) and, simultaneously, the term $k_3 u v$ using the second equation of (3.6). We then obtain a simpler equation for the projection of the momenta onto the x axis

$$\beta_1 u^2 + x\xi'(x) u \frac{\partial u}{\partial\xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial\xi}\right)\left(\frac{\partial u}{\partial\eta} + k_6 u\right) =$$
$$= -\frac{V_\infty^2}{\rho u_*^2} x\xi'(x)\frac{\partial p}{\partial\xi} + \frac{\partial}{\partial\eta}\left(\frac{1}{\mathrm{Re}^* \Delta^2}\tau_{\xi\eta}\right) + \frac{k_6}{\mathrm{Re}^*\Delta^2}\tau_{\xi\eta} + \Phi_1 + \Phi_2 \tag{3.14}$$

where

$$\Phi_1 = \varepsilon_1 \tau'_{\xi\xi} - \varepsilon_2\tau_{\zeta\zeta} - \rho k_1 k_8\left[\frac{\partial}{\partial\eta}\left(\varepsilon_3\tau_{\eta\eta}\right) + \varepsilon_3\tau'_{\xi\eta} - \varepsilon_4\tau_{\xi\xi} - \varepsilon_5\tau_{\zeta\zeta}\right]$$

$$\Phi_2 = \rho k_1 k_8\left[\beta_2 u v + x\xi'(x)\frac{\partial v}{\partial\xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial\xi}\right)\frac{\partial v}{\partial\eta}\right] \tag{3.15}$$

which is more convenient for the subsequent analysis and solution.

Hence, the final system of Navier–Stokes equations in the variables ξ, η will consist of the six equations (1.5), (3.6), (3.10), (3.11) and (3.14) for determining the six functions: $u, f, v; p, \rho, T$ (or H).

The boundary conditions on the wall (2.1) in the new variables will be

$$u(\xi, 0) = 0, \quad v(\xi, 0) = f(\xi, 0) = 0$$

$$\left.\frac{\mu\rho}{\sigma\Delta}\frac{\partial H}{\partial\eta}\right|_{\eta=0} = \varepsilon\sigma_B T_w^4(\xi, 0) \quad \text{or} \quad T(\xi, 0) = T_w(\xi) \tag{3.16}$$

The boundary conditions in the free stream (2.2) for solving the full system of Navier–Stokes equations (1.5), (3.6), (3.10), (3.11) and (3.14) will be

$$u(\xi,\infty)=\frac{v_{1\infty}}{u_*}=\frac{V_\infty\cos\alpha}{u_*},\quad v(\xi,\infty)=\frac{v_{2\infty}}{v_*}=-\frac{V_\infty\sin\alpha}{v_*}$$

$$\rho(\xi,\infty)=1,\quad p(\xi,\infty)=\frac{1}{\gamma M_\infty^2},\quad H(\xi,\infty)=H_\infty\equiv h_\infty+\frac{V_\infty^2}{2} \tag{3.17}$$

We shall write the boundary conditions on the shock wave (on the separating surface $y_s=y_s(x)$) (2.8), (2.9) and (2.11) for the solution of the simplified Navier–Stokes equations, taking account of (2.10)–(2.12) and (3.7), in the form

$$\begin{aligned}
&u(\xi,\eta_s)=\frac{u_i}{u_*}-m_{4s}\frac{\cos^3\beta_s}{\sin\beta}\left[\frac{\tau_{\xi\eta}}{\mathrm{Re}^*\Delta^2}(1-\mathrm{tg}^2\beta_s)-\varepsilon_6(\tau_{\eta\eta}-\tau_{\xi\xi})\mathrm{tg}\,\beta_s\right]_s\\
&v(\xi,\eta_s)=\frac{u_*}{v_*}u(\xi,\eta_s)-k\frac{V_\infty}{v_*}\frac{\sin\beta}{\cos\beta_s}\\
&p(\xi,\eta_s)=\frac{1}{\gamma M_\infty^2}+(1-k)\sin^2\beta+\\
&+m_{5s}\cos^2\beta_s\left[-\frac{2\tau_{\xi\eta}}{\mathrm{Re}^*\Delta^2}\mathrm{tg}\,\beta_s+\varepsilon_6(\tau_{\eta\eta}+\tau_{\xi\xi}\,\mathrm{tg}^2\beta_s)\right]_s\\
&H(\xi,\eta_s)=H_\infty-m_{4s}\frac{\cos\beta_s}{\sin\beta}\left(\frac{1}{\mathrm{Re}^*\Delta^2\sigma}Y-\varepsilon_6X\right)_s\\
&k=\frac{1}{\rho_s}=\frac{R_A}{V_\infty^2}\frac{T_s}{p_s}=\frac{\gamma-1}{2\gamma p_s}\frac{T_s}{T_0}
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
&m_4=\frac{u_*\Delta}{V_\infty xH_1\bar r^\nu}\\
&m_5=\frac{u_*^2\Delta}{V_\infty^2 xH\bar r^\nu}=m_4\frac{u_*}{V_\infty},\quad \varepsilon_6=\frac{\mu\bar r^\nu}{\rho_\infty u_*\Delta},\quad \eta_s=\frac{1}{\Delta}\int_0^{y_s(x)}\rho\bar r^\nu dy
\end{aligned} \tag{3.19}$$

The subscript s in (3.18) and (3.19) denotes that the corresponding expressions must be calculated at $y_s=y_s(x)$.

For the actual solution of the problems it is convenient to replace the second condition of (3.18) by an equivalent condition, which is imposed on the function $f(\xi,\eta)$. From the second equation of (3.6), written on the shock wave, we obtain

$$f_s(\xi)\equiv f(\xi,\eta_s)=\frac{V_\infty r_w\bar r_s^{\nu+1}}{(\nu+1)u_*\Delta} \tag{3.20}$$

It can also be shown that relation (3.20) is a consequence of the law of conservation of the mass of the gas which flows across the closed contour $ONQO_s$ (Fig. 1), written in ξ, η variables.

The relation between the function $\Delta(x)$ and the physical standoff distance of the shock wave follows from the second equation of (3.2) written when $y_s=y_s(x)$

$$y_s\frac{\cos\alpha}{r_w}=\left[1+(\nu+1)\Delta\frac{\cos\alpha}{r_w}\int_0^{\eta_s}\frac{1}{\rho}d\eta\right]^{\frac{1}{\nu+1}}-1 \tag{3.21}$$

Eliminating $r_s^{-\nu+1}$ from (3.2) and (3.21), we obtain, instead of (3.20), the condition for the reduced stream function on the shock wave

$$f_s(\xi) \equiv f(\xi, \eta_s) = \frac{V_\infty r_w}{(\nu+1)u_*\Delta}\left[1 + (\nu+1)\Delta\frac{\cos\alpha}{r_w}\int_0^{\eta_s}\frac{1}{\rho}d\eta\right] \tag{3.22}$$

which does not contain the physical standoff distance of the shock wave $y_s = y_s(x)$. This condition provides an explicit relation between the function Δ and the function f on the shock wave. Formula (3.21) takes a quite simple form in the plane case ($\nu = 0$)

$$y_s(x) = \Delta\int_0^{\eta_s}\frac{1}{\rho}d\eta \tag{3.23}$$

The friction coefficient (2.14), the heat transfer coefficient (2.15) and the drag coefficient (2.16) in the variables (3.1) will be

$$\begin{aligned} C_f &= \frac{2u_*}{V_\infty K_w \Delta}\frac{\partial u(\xi,0)}{\partial\eta} = \frac{2u_*}{V_\infty K_* \bar{\Delta}}\left(l\frac{\partial u}{\partial \eta}\right)_w \\ C_H &= \frac{1}{K_w\Delta\sigma}\frac{1}{H_\infty - h_w}\frac{\partial H(\xi,0)}{\partial\eta} = \frac{1}{K_*\bar{\Delta}(H_\infty - h_w)}\left(\frac{l}{\sigma}\frac{\partial H}{\partial\eta}\right)_w \\ C_D &= C_{Dp} + C_{Df} \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} C_{Dp} &= \frac{4}{r_w^{\nu+1}(x)}\int_0^x\left(p_w - \frac{1}{\gamma M_\infty^2}\right)r_w^\nu \sin\alpha dx = \frac{2}{r_w^{\nu+1}(x)}\int_0^x C_p r_w^\nu \sin\alpha dx \\ C_{Df} &= \frac{4}{r_w^{\nu+1}(x)}\int_0^x \frac{u_*\cos\alpha}{V_\infty K_w\Delta}\frac{\partial u(\xi,0)}{\partial\eta}r_w^\nu dx = \frac{2}{r_w^{\nu+1}(x)}\int_0^x C_f r_w^\nu \cos\alpha dx \\ K_w &= \frac{\rho_\infty V_\infty}{(\mu\rho)_w}, \quad K_* = \frac{\rho_\infty V_\infty R_0}{\mu_*\rho_*}, \quad l = \frac{\mu\rho}{\mu_*\rho_*}, \quad \bar{\Delta} = \frac{\Delta}{R_0} \end{aligned} \tag{3.25}$$

Here, μ_* and ρ_* are the coefficient viscosity and the dimensionless density taken in some cross-section $\eta = \eta_*$ of the shock layer and R_0 is a characteristic linear scale of the problem, such as the radius of curvature at the body vertex, for example. The overall drag C_D is represented in the form of the sum of the inviscid and viscous drag coefficients.

4. AN ESTIMATE OF THE ORDER OF THE COEFFICIENTS OF THE DIMENSIONLESS NAVIER–STOKES EQUATIONS

In the case of supersonic and hypersonic flows around blunt bodies, the ratio of the densities up to and immediately after the leading shock wave and the Reynolds number are the main governing parameters. In the case of a perfect diatomic gas with constant heat capacities at high Mach and Reynolds numbers, the parameter k can be explicitly calculated, and its greatest value on the shock layer is equal to $k = (\gamma - 1)/(\gamma + 1)$. In this case, the parameter k becomes governing. In the case of a gas, when account is taken of the physicochemical processes of viscosity and thermal conductivity in the shock layer, its value changes from 0.05 to 0.2 while, at meteor velocities of the order of several tens of kilometres per second, k becomes even smaller [30]. In the case of the motion of bodies in the Earth's atmosphere at Mach numbers $M_\infty \geqslant 6$, this parameter defines the shock-layer thickness $y_s \sim kR_0$ [30] (also, see (3.23)) and will subsequently vary as the governing parameter from one to zero. The Reynolds number is also a governing parameter and characterizes the thickness of the boundary layer and of the shock wave structure and will vary from infinity to zero.

We will now estimate the orders of magnitude of the coefficients in the Navier–Stokes equations (3.6), (3.10), (3.11) and (3.14) as a function of the parameter k and the number Re.

From expressions (3.13), for an arbitrary choice of the characteristic velocities u_* and υ_* and the scale function $\Delta(x)$, for fairly smooth bodies we have

$$\beta_0 \sim \beta_1 \sim \beta_2 \sim 1, \ m_1 \sim m_2 \sim m_3 \sim n_1 \sim 1 \tag{4.1}$$

We determine the order of the function $\Delta(x)$ in (3.1) from the condition that the variable η both in the scale of the boundary-layer thickness as well as in the scale of the shock-layer thickness must be of the order of unity. The estimates

$$y \sim k\Delta, \ \ y_s \sim k\Delta \tag{4.2}$$

will then subsequently hold for all of the hydrodynamic models.

So far, estimates (4.2) do not contain the final information on the order of magnitude of the thicknesses of the flow domains around the body, since the order of magnitude of the function $\Delta(x)$ has still not been determined. In fact, the order of $\Delta(x)$ will be determined starting out from the transverse scale of some flow domain which is being considered. The remaining coefficients in the Navier–Stokes equations (3.6), (3.10), (3.11) and (3.14) will be of a different order of magnitude in the case of the inviscid part of the shock layer and boundary layer since the parameters (functions) $u_*(x)$, $\upsilon_*(x)$ and $\Delta(x)$, which govern the characteristic velocities in these domains and their transverse dimensions, appear in these equations.

We will first obtain the orders of magnitude of these coefficients in the case of a shock layer, starting out from the scales

$$u_* \sim V_\infty, \ \ \upsilon_* \sim \upsilon_i \sim kV_\infty \tag{4.3}$$

which follow from boundary conditions (3.18) and (2.10).

We select $\Delta(x)$ from the condition for normalizing the standoff distance wave in the variables (3.1)

$$\Delta(x) = \int_0^{y_s(x)} \rho \bar{r}^\nu dy \sim R_0 \tag{4.4}$$

The order of $\Delta(x)$ in (4.4) is obtained taking account of the first equation of (3.6), the scale (4.3) for u_* and relation (3.20), while noting that $\eta_s = 1$ in this relation. Then, in the variables (3.1), when account is taken of (4.4), the physical domain of integration over the shock layer is transformed into the half-strip: $0 \leqslant \xi(x) \leqslant \xi_*$, $0 < \eta < 1$, that is, $\eta \sim 1$ from which, bearing in mind that $H_1 \sim \bar{r} \sim 1 + k \sim 1$, we obtain

$$k_1 \sim k_2 \sim k_3 \sim k_4 \sim k_5 \sim k_6 \sim k_7 \sim k, \ \ k_8 \sim k^2 \tag{4.5}$$

for the coefficients k_i ($i = 1, \dots 8$).

From (3.13) and (4.3), we find the order of magnitude of the coefficients ε_i ($i = 1, \dots 5$) in the shock layer

$$\varepsilon_1 \sim \varepsilon_2 \sim k/\mathrm{Re}, \ \ \varepsilon_3 \sim 1/(k\,\mathrm{Re}), \ \ \varepsilon_4 \sim \varepsilon_5 \sim 1/\mathrm{Re} \tag{4.6}$$

where

$$\mathrm{Re} = \frac{\rho_\infty u_* R_0}{\mu} \sim \frac{\rho_\infty V_\infty R_0}{\mu(T_0)} \tag{4.7}$$

Estimates for the dimensionless components of the viscous stress tensor τ_{ij} ($i, j = \xi, \tau, \zeta$) and the fluxes X and Y (3.12) will be

$$\begin{gathered}\tau_{\xi\xi} \sim \tau_{\eta\eta} \sim \tau_{\zeta\zeta} \sim \tau_{\xi\eta} \sim \tau'_{\xi\xi} \sim \tau'_{\xi\eta} \sim 1 \\ X \sim X' \sim Y \sim \Delta H, \ \ \Delta H = H_\infty - h_w \sim V_\infty^2\end{gathered} \tag{4.8}$$

The order of the coefficients in (3.9), for the conditions on the shock wave (3.18), will be

$$m_{4s} \sim m_{5s} \sim 1, \quad \varepsilon_{6s} \sim \mathrm{Re}_s^{-1} \tag{4.9}$$

where

$$\mathrm{Re}_s = \frac{\rho_\infty V_\infty R_0}{\mu_s} \tag{4.10}$$

The boundary conditions on the wall (3.16) and in the free stream (3.17) do not change the estimates presented above.

We will next consider all possible limiting cases when $0 < k < 1$ over the whole range of Reynolds numbers from infinity to zero.

5. THE BOUNDARY-LAYER EQUATIONS (Re $\to \infty$, $k \sim 1$ AND TERMS OF THE ORDER OF UNITY ARE RETAINED IN THE NAVIER–STOKES EQUATIONS)

Suppose that $\mathrm{Re}_s = \rho_\infty V_\infty R_0/\mu_s \to \infty$, where μ_s is the value of the viscosity immediately behind the shock wave. The parameter $k \sim 1$. Then, the estimate ($u \sim 1, f \sim 1, u_* \sim V_\infty$)

$$\mathrm{Re}^* \Delta^2 = \frac{\rho_\infty u_* \Delta^2}{\mu \rho x H_1 \bar{r}^{2\nu}} \sim K\bar{\Delta}^2 \sim 1 \tag{5.1}$$

where

$$K = \frac{\rho_\infty V_\infty R_0}{\mu\rho} \sim k\,\mathrm{Re}, \quad \mathrm{Re} = \frac{\rho_\infty V_\infty R_0}{\mu} \tag{5.2}$$

follows from the assumption that the viscous terms in Eq. (3.14) when $\mathrm{Re}_s \to \infty$ are only of the same order of magnitude as the inertial terms in a thin layer around the wall (the boundary layer) $y_\delta \ll R_0$ (Prandtl's principal hypothesis when deriving the boundary-layer equations).

It follows from this that

$$\bar{\Delta} \sim K^{-1/2} \sim \mathrm{Re}^{-1/2} \tag{5.3}$$

Then, from (4.2), we obtain the well-known result for the boundary-layer thickness

$$y / R_0 \sim y_\delta / R_0 \sim \sqrt{k / \mathrm{Re}} \sim \mathrm{Re}^{-1/2} \tag{5.4}$$

It follows from the second equation of (3.6), which is a consequence of the equation of continuity (1.1), that $k_2 \sim 1$, from which we also obtain the well-known result for the order of magnitude of the normal component of the velocity

$$v_* / u_* \sim v_* / V_\infty \sim v / V_\infty \sim \bar{\Delta} \sim \mathrm{Re}^{-1/2} \tag{5.5}$$

Taking (5.3) and (5.4) into account, we conclude that the remaining coefficients in (3.13) will be of the following orders of magnitude ($k \sim 1$)

$$k_1 \sim 1, \quad k_3 \sim k_4 \sim k_6 \sim k_7 \sim \mathrm{Re}^{-1/2}, \quad k_5 \sim k_8 \sim \mathrm{Re}^{-1} \tag{5.6}$$

$$\varepsilon_1 \sim \varepsilon_2 \sim \mathrm{Re}^{-1}, \quad \varepsilon_3 \sim 1, \quad \varepsilon_4 \sim \varepsilon_5 \sim \mathrm{Re}^{-1/2} \tag{5.7}$$

By virtue of the fact that the operator D, which is applied to the required functions in (3.12), is of the order of unity, estimates (4.8) follow, which hold when $\mathrm{Re} \to \infty$, in view of (4.1). From this, bearing (5.6) and (5.7) in mind, we find (see (3.15))

$$\Phi_1 \sim \Phi_2 \sim \mathrm{Re}^{-1}$$

On letting Re tend to infinity and retaining only terms of the order of unity in Eqs (3.14), (3.10) and

(3.11) (first-order or classical boundary-layer theory), we obtain the boundary-layer equations

$$\beta_1 u^2 + x\xi'(x)u\frac{\partial u}{\partial \xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial u}{\partial \eta} =$$

$$= -\frac{V_\infty^2}{\rho u_*^2}x\xi'(x)\frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \eta}\left(\frac{1}{\mathrm{Re}^* \Delta^2}\frac{\partial u}{\partial \eta}\right) + O(\mathrm{Re}^{-1/2}) \tag{5.8}$$

$$\frac{\partial p}{\partial \eta} = O(\mathrm{Re}^{-1/2}) \tag{5.9}$$

$$x\xi'(x)u\frac{\partial H}{\partial \xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial H}{\partial \eta} =$$

$$= \frac{\partial}{\partial \eta}\left\{\frac{1}{\mathrm{Re}^* \Delta^2 \sigma}\left[\frac{\partial H}{\partial \eta} + (\sigma - 1)u_*^2\frac{\partial}{\partial \eta}\left(\frac{u^2}{2}\right)\right]\right\} + O(\mathrm{Re}^{-1/2}) \tag{5.10}$$

It is necessary to add system (3.6) here which, neglecting terms of the order of Re^{-1} in the second equation in which the leading terms are of the order of $\mathrm{Re}^{-1/2}$, will be

$$u = \frac{\partial f}{\partial \eta}, \quad v_2 = -v_1\xi'(x)\frac{\partial y}{\partial \xi} - \frac{\Delta u_*}{\rho x}\left[\beta_0 f + x\xi'(x)\frac{\partial f}{\partial \xi}\right] \tag{5.11}$$

In system (5.8)–(5.11), the two functions $\xi(x)$ and $\Delta(x)$ remain arbitrary. On multiplying equations (5.8) and (5.10) by $c\xi/(x\xi'(x))$ an selecting the functions $\xi(x)$ and $\Delta(x)$ from the conditions (c is a required constant)

$$\frac{c\xi}{x\xi'(x)}\beta_0 = 1, \quad \frac{c\xi}{x\xi'(x)}\frac{1}{\mathrm{Re}^* \Delta^2} = l, \quad l = \frac{\mu\rho}{\mu_*\rho_*} = \frac{\mu\rho_\infty\rho}{\mu_*\rho_\infty\rho_*}$$

we find the Dorodnitsyn–Lees transform

$$\xi(x) = \int_0^x \mu_*\rho_\infty\rho_* u_* r_w^{2\nu}dx, \quad \eta = \frac{\rho_\infty u_* r_w^\nu}{\sqrt{2\xi}}\int_0^y \rho dy = \sqrt{\frac{K_*}{2\bar{X}}}\int_0^{\bar{y}} \rho d\bar{y} \tag{5.12}$$

$$\Delta = \frac{\sqrt{2\xi}}{\rho_\infty u_* r_w^\nu} = R_0\sqrt{\frac{2\bar{X}}{K_{**}}}, \quad b = \sqrt{2\xi}, \quad c = 2 \tag{5.13}$$

where

$$\bar{X} = \frac{\xi}{\xi'(x)R_0}, \quad K_{**} = \frac{\rho_\infty u_* R_0}{\mu_*\rho_*} = \frac{u_*}{V_\infty}K_*, \quad \bar{y} = \frac{y}{R_0} \tag{5.14}$$

The asterisk refers to the conditions in any characteristic cross-section $\eta = \eta_*$ of the boundary layer and the functions $X = \bar{X}R_0$ may be called the effective length of the plate, since the variable η in (5.12) becomes the self-similar Blasius variable if X is replaced in it by x, that is, the distance from the leading edge of the plate.

In the variables ξ, η of (5.12) Eqs (5.8)–(5.11) take the well-known form

$$u = \frac{\partial f}{\partial \eta}, \quad v_2 = -\frac{v_1}{X}2\xi\frac{\partial y}{\partial \xi} - \frac{u_*}{\rho\sqrt{2\bar{X}K_{**}}}\left(f + 2\xi\frac{\partial f}{\partial \xi}\right) \tag{5.15}$$

$$2\xi u\frac{\partial u}{\partial \xi} - \left(f + 2\xi\frac{\partial f}{\partial \xi}\right)\frac{\partial u}{\partial \eta} = -\frac{V_\infty^2}{\rho u_*^2}2\xi\frac{\partial p}{\partial \xi} - \beta_1 u^2 + \frac{\partial}{\partial \eta}\left(l\frac{\partial u}{\partial \eta}\right) \tag{5.16}$$

$$2\xi u\frac{\partial H}{\partial \xi}-\left(f+2\xi\frac{\partial f}{\partial \xi}\right)\frac{\partial H}{\partial \eta}=\frac{\partial}{\partial \eta}\left\{\frac{l}{\sigma}\left[\frac{\partial H}{\partial \eta}+(\sigma-1)u_*^2\frac{\partial}{\partial \eta}\left(\frac{u^2}{2}\right)\right]\right\} \tag{5.17}$$

The solution of these equations by the method of matched asymptotic expansions, retaining terms of the order of unity, must be coupled with the solution of Euler's equations (the Navier–Stokes equations which are treated outside the boundary layer when $\mathrm{Re} \to \infty$, retaining terms $O(1)$) when $y \to 0$. The equation for the momenta along the x axis in the case of an inviscid gas written on the body (when $y \to 0$) gives

$$-\frac{V_\infty}{\rho_e u_e^2}2\xi\frac{\partial p}{\partial \xi}=\beta,\quad \beta=2\frac{d\ln u_e}{d\ln \xi} \tag{5.18}$$

where $u_e \equiv v_{1e}$ is the velocity of inviscid flow over the body. If we now take $u_* = u_e$ as the characteristic velocity and use (5.18) in order to eliminate $\partial p/(\partial \xi)$ from Eq. (5.16), it takes the final form which is customarily employed when solving boundary-layer problems

$$2\xi u\frac{\partial u}{\partial \xi}-\left(f+2\xi\frac{\partial f}{\partial \xi}\right)\frac{\partial u}{\partial \eta}=\beta\left(\frac{\rho_e}{\rho}-u^2\right)+\frac{\partial}{\partial \eta}\left(l\frac{\partial u}{\partial \eta}\right) \tag{5.19}$$

Conditions (3.16) remain as the boundary conditions on the body for system (5.15), (5.17), (5.19). On the boundary layer edge when $\eta \to \infty$, these conditions will be

$$u(\xi,\infty)=1,\quad H(\xi,\infty)=H_e=H_\infty \tag{5.20}$$

Numerous approximate and efficient numerical methods of solution have been developed for this parabolic problem. After it has been solved, the friction coefficient and the heat transfer coefficient (3.24) will be calculated, taking account of (5.13) from the formulae

$$C_f=\frac{\tau_{xyw}}{\frac{1}{2}\rho_\infty V_\infty^2}=\left(\frac{u_e}{V_\infty}\right)^2\sqrt{\frac{2}{K_{**}X}}\left(l\frac{\partial u}{\partial \eta}\right)_w,\quad K_{**}=\frac{\rho_\infty u_* R_0}{\mu_*\rho_*}=\frac{\rho_\infty u_e R_0}{\mu_*\rho_*} \tag{5.21}$$

$$C_H=\frac{q_w}{\rho_\infty V_\infty(H_\infty-h_w)}=\frac{u_e}{V_\infty}\frac{1}{\sqrt{2K_{**}X}}\left(\frac{l}{\sigma}\frac{\partial H}{\partial \eta}\right)_w\frac{1}{H_e-h_w} \tag{5.22}$$

6. THE VISCOUS SHOCK-LAYER EQUATIONS ($\mathrm{Re} \to \infty$, $k \sim 1$ AND TERMS $O(1)$ AND $O(\mathrm{Re}^{-1/2})$ ARE RETAINED IN THE NAVIER–STOKES EQUATIONS)

Letting the Reynolds number in the Navier–Stokes equations (3.6), (3.10), (3.11) and (3.14) tend to infinity, retaining terms of the order of $\mathrm{Re}^{-1/2}$ in them and taking account of the fact here that, in the whole of the shock layer including the boundary layer, estimates (4.1) and (4.2) remain valid and that estimates (5.3)–(5.7) and (4.8) still hold in the boundary layer, we obtain the following systems of equations

$$\frac{\partial f}{\partial \eta}=u,\quad \rho(k_1u+k_2v)=-\left(\beta_0 f+x\xi'(x)\frac{\partial f}{\partial \xi}\right) \tag{6.1}$$

$$\beta_1u^2+x\xi'(x)u\frac{\partial u}{\partial \xi}-\left(\beta_0 f+x\xi'(x)\frac{\partial f}{\partial \xi}\right)\left(\frac{\partial u}{\partial \eta}+k_6u\right)=$$
$$=-\frac{V_\infty^2}{\rho u_*^2}x\xi'(x)\frac{\partial p}{\partial \xi}+\frac{\partial}{\partial \eta}\left[\frac{1}{\mathrm{Re}^*\Delta^2}\left(\frac{\partial u}{\partial \eta}-k_6u\right)\right]+\frac{k_6}{\mathrm{Re}^*\Delta^2}\frac{\partial u}{\partial \eta}+$$
$$+\rho k_1k_8\left[\beta_2uv+x\xi'(x)u\frac{\partial u}{\partial \xi}-\left(\beta_0 f+x\xi'(x)\frac{\partial f}{\partial \xi}\right)\frac{\partial v}{\partial \eta}\right]+O(\mathrm{Re}^{-1}) \tag{6.2}$$

$$\frac{u_*^2 \Delta}{V_\infty^2 H_1 \bar{r}^\nu R} u^2 = \frac{\partial p}{\partial \eta} + O(\mathrm{Re}^{-1}) \tag{6.3}$$

$$x\xi'(x) u \frac{\partial H}{\partial \xi} - \left(\beta_0 f + x\xi'(x) \frac{\partial f}{\partial \xi} \right) \frac{\partial H}{\partial \eta} =$$

$$= \frac{\partial}{\partial \eta} \left\{ \frac{1}{\mathrm{Re}^* \Delta^2 \sigma} \left[\frac{\partial H}{\partial \eta} + u_*^2 \left((\sigma - 1) \frac{\partial}{\partial \eta} \left(\frac{u^2}{2} \right) - \sigma k_6 u^2 \right) \right] \right\} + O(\mathrm{Re}^{-1}) \tag{6.4}$$

Note that the last term in square brackets on the right-hand side of (6.2) is of the order of Re^{-1} according to estimates (5.6) and must be omitted. However, it remains available for future use in order to obtain VSL equations which are uniformly valid over the whole of the shock wave (see below).

The boundary conditions on the wall for this system remain as previously (3.16). An important conclusion follows from Eq. (6.3), that is, the pressure across the boundary layer in the second approximation can only change due to centrifugal forces. In order to determine this pressure, using an equation of the first order in the coordinate η (6.3), a single boundary condition on the boundary layer edge is required (the pressure on the wall is unknown). The solution of Euler's equations on the wall cannot be taken as the outer boundary conditions for Eqs (6.1)–(6.4) since, in the second approximation, it (the outer solution) will be perturbed by the viscosity and thermal conductivity and, in particular, by the displacement thickness of the boundary layer and the effect of the vorticity of inviscid flow through coupling with the boundary-layer solution in the second approximation [6]. Hence, in order to obtain a closed problem in the second approximation of boundary-layer theory (as an alternative to the asymptotic method of matched outer and inner expansions) we will try to obtain a composite system of equations which uniformly and simultaneously describes both the inviscid flow domain and the viscous flow domain over the whole of the shock layer with boundary conditions on the dividing line $y_s = y_s(x)$ with an accuracy up to terms of the order of Re^{-1}. For this purpose, it is necessary to estimate and keep the $O(1)$ and $O(\mathrm{Re}^{-1/2})$ terms in the outer inviscid domain of the shock layer when $\mathrm{Re} \to \infty$ in the system of Navier–Stokes equations (3.6), (3.10), (3.11) and (3.14) and boundary conditions (3.18). In this case, unlike when estimating the boundary-layer terms, we shall start out from estimates (4.5), (4.6) and (4.8), which hold in the inviscid part of the shock layer. We then obtain

$$\Phi_1 \sim \mathrm{Re}^{-1}, \quad \Phi_2 \sim 1 \tag{6.5}$$

Hence, retaining the last term on the right-hand side of Eq. (6.2) is justified. When $\mathrm{Re} \to \infty$, we then obtain a system of Euler equations in the variables ξ and η from (3.9)–(3.11). Equations (6.1) must be added to this system. Consequently, taking account of terms of the order $\mathrm{Re}^{-1/2}$ does not add any additional terms to Euler's equations in the inviscid domain of the shock layer, unlike the system in the viscous domain (6.1)–(6.4) which, compared with the first approximation of boundary-layer equations, contains additional terms of the order of $\mathrm{Re}^{-1/2}$. Hence, the effect of viscosity and thermal conductivity on an inviscid flow in a shock layer is achieved through the condition for the matching solution of Euler's equations when $y \to 0$ with the solution of the boundary-layer equations in the second approximation.

In order to obtain a composite system of equations which describes the flow in the whole of the shock layer, taking account of second-order terms, it is now necessary to replace the simplified equation (6.3) in system (6.1)–(6.4) by the complete equation of motion of the inviscid fluid, that is

$$\beta_2 uv - \frac{1}{k_4} u^2 + x\xi'(x) u \frac{\partial v}{\partial \xi} - \left(\beta_0 f + x\xi'(x) \frac{\partial f}{\partial \xi} \right) \frac{\partial v}{\partial \eta} = -\frac{1}{k_5} \frac{\partial p}{\partial \eta} + O(\mathrm{Re}^{-1}) \tag{6.6}$$

which, on the right-hand side, contains terms of the order of unity in the inviscid domain which become terms of the order of Re^{-1} and $\mathrm{Re}^{-1/2}$ (the second term on the right) in the viscous domain. So, the system of VSL equations (6.1), (6.2), (6.6) and (6.4) will be the composite system of equations which takes account of all second-order effects. Note that, in the VSL system, the direct influence of viscosity in the equation for the momenta along the normal (6.6) is a third-order effect in Re^{-1} although the second-order effects associated with curvature occur directly in the coefficients of the equation. In the equation for the x axis projection of the momenta (6.2), the effect of viscosity already manifests itself in the first and second order. Nevertheless, the VSL equations are uniformly useful up to terms

$O(\mathrm{Re}^{-1/2})$ inclusive over the whole of the shock layer. This is an extremely important and fundamental simplification of the Navier–Stokes equations which leads to the VSL equations if it is remembered that the Navier–Stokes equations themselves are only asymptotically true when account is taken of terms of the order of those terms that were left in the system of equations (6.1), (6.2), (6.4) and (6.6).

We will now give an asymptotic simplification of the boundary conditions in the shock wave in the case of the VSL equations. Bearing in mind that, in the second order, viscosity manifests itself over the whole of the shock layer, it is necessary to use estimate (5.3) for Δ. Then, for the coefficients (3.10) in conditions (3.18), we obtain the estimates

$$\frac{m_4}{\mathrm{Re}^*\,\Delta^2}=\frac{\mu\rho\bar{r}^{\nu}}{\rho_\infty V_\infty\Delta}=\frac{\bar{r}^{\nu}}{K\bar{\Delta}}\sim K^{-1/2},\quad m_4\varepsilon_6=\frac{\mu}{\rho_\infty V_\infty x H_1}\sim \mathrm{Re}^{-1}$$
$$\frac{m_5}{\mathrm{Re}^*\,\Delta^2}=\frac{u_*}{V_\infty}\frac{\mu\rho\bar{r}^{\nu}}{\rho_\infty V_\infty\Delta}\sim K^{-1/2},\quad m_5\varepsilon_6=\frac{\mu u_*}{\rho_\infty V_\infty x H_1}\sim \mathrm{Re}^{-1} \tag{6.7}$$

On retaining terms $O(1)$ and $O(\mathrm{Re}^{-1/2})$ in relations (3.18) and taking account of (6.7), we obtain the so-called generalized Rankine–Hugoniot conditions

$$u(\xi,\eta_s)=\frac{u_i}{u_*}-\left[\frac{\bar{r}^{\nu}\cos^3\beta_s(1-\mathrm{tg}^2\beta_s)}{K\bar{\Delta}\sin\beta}\frac{\partial u}{\partial\eta}\right]_s$$
$$p(\xi,\eta_s)=\frac{1}{\gamma M_\infty^2}+(1-k)\sin^2\beta-\left(\frac{u_*}{V_\infty}\frac{\bar{r}^{\nu}2\cos^2\beta_s\,\mathrm{tg}\beta_s}{K\bar{\Delta}}\frac{\partial u}{\partial\eta}\right)_s \tag{6.8}$$
$$H(\xi,\eta_s)=H_\infty-\frac{\bar{r}^{\nu}\cos\beta_s}{K\bar{\Delta}\sin\beta}\left[\frac{1}{\sigma}\frac{\partial H}{\partial\eta}+u_*^2\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial\eta}\left(\frac{u^2}{2}\right)\right]$$
$$K=\frac{\rho_\infty V_\infty R_0}{\mu\rho}$$

These conditions do not take account of the fact that the derivative $(\partial v)/(\partial\eta)$ becomes large on approaching the shock wave. Retaining this derivative in relations (3.18), we obtain conditions on the required shock wave which are more complex than (6.8)

$$u(\xi,\eta_s)=\frac{u_i}{u_*}-\frac{\bar{r}^{\nu}\cos^3\beta_s(1-\mathrm{tg}^2\beta_s)}{K\bar{\Delta}\sin\beta}\frac{\partial u}{\partial\eta}-\frac{2v_*}{u_*}\frac{\bar{r}^{\nu}\cos^3\beta_s\,\mathrm{tg}\beta_s}{K\bar{\Delta}\sin\beta}\frac{\partial v}{\partial\eta}$$
$$p(\xi,\eta_s)=\frac{1}{\gamma M_\infty^2}+(1-k)\sin^2\beta-\frac{u_*}{V_\infty}\frac{2\bar{r}^{\nu}\cos^2\beta_s\,\mathrm{tg}\beta_s}{K\bar{\Delta}}\frac{\partial u}{\partial\eta}+$$
$$+\frac{\rho v_*\bar{r}^{\nu}\cos^2\beta_s}{V_\infty K\bar{\Delta}}\left[\zeta+\frac{4}{3}+\left(\zeta-\frac{2}{3}\right)\mathrm{tg}^2\beta_s\right]\frac{\partial v}{\partial\eta} \tag{6.9}$$
$$H(\xi,\eta_s)=H_\infty-\frac{\bar{r}^{\nu}\cos\beta_s}{K\bar{\Delta}\sin\beta}\left[\frac{1}{\sigma}\frac{\partial H}{\partial\eta}+u_*^2\left(1-\frac{1}{\sigma}\right)\frac{\partial}{\partial\eta}\left(\frac{u^2}{2}\right)+\right.$$
$$\left.+\left(\zeta+\frac{4}{3}-\frac{1}{\sigma}\right)v_*^2\frac{\partial}{\partial\eta}\left(\frac{v^2}{2}\right)\right]+$$
$$+\frac{\bar{r}^{\nu}u_*v_*\cos\beta_s\,\mathrm{tg}\beta_s}{K\bar{\Delta}\sin\beta}\left[\left(\zeta-\frac{2}{3}\right)u-\frac{v_*\xi}{X u_* H_1}\frac{\partial y}{\partial\xi}v\right]\frac{\partial v}{\partial\eta}$$

If the parameter k is not assumed to be small, the additional (third) terms in conditions (6.9) are of the same order $K^{-1/2}$ as the "viscous" terms in (6.8). The second condition of (3.18) is replaced by the equivalent condition (3.20) or (3.22). The standoff distance of the shock wave will then be found from (3.20) or (3.21).

We shall write the VSL equations in the boundary-layer variables (5.12); the contribution to them from effects of the second approximation, compared with boundary-layer equations (5.15), (5.17) and (5.19) written in the same variables, will then be clearly seen.

Omitting the calculations, which are similar to those presented in Section 5, we find

$$u=\frac{\partial f}{\partial\eta},\quad v_2=\frac{\xi}{XH_1}\frac{\partial y}{\partial\xi}u_*u-\frac{u_*\left(f+2\xi\frac{\partial f}{\partial\xi}\right)}{\rho H_1\bar{r}^{\nu}\sqrt{2\bar{X}K_*}} \tag{6.10}$$

$$\beta_{1\xi}u^2+2\xi u\frac{\partial u}{\partial\xi}-\left(f+2\xi\frac{\partial f}{\partial\xi}\right)\left(\frac{\partial u}{\partial\eta}+k_6u\right)=$$
$$=-\frac{V_\infty^2}{\rho u_*^2}2\xi\frac{\partial p}{\partial\xi}+\frac{\partial}{\partial\eta}(\mathcal{L}\tau_{\xi\eta})+a\frac{\partial u}{\partial\eta}+c\Phi \tag{6.11}$$

$$\frac{\partial p}{\partial\eta}=du^2-e\Phi \tag{6.12}$$

$$2\xi u\frac{\partial H}{\partial\xi}-\left(f-2\xi\frac{\partial f}{\partial\xi}\right)\frac{\partial H}{\partial\eta}=\frac{\partial}{\partial\eta}(\mathcal{L}Y) \tag{6.13}$$

where

$$\Phi=\beta_{2\xi}uv+2\xi u\frac{\partial v}{\partial\xi}-\left(f+2\xi\frac{\partial f}{\partial\xi}\right)\frac{\partial v}{\partial\eta}$$

$$Y=\frac{\partial H}{\partial\eta}+u_*^2\left[(\sigma-1)\frac{\partial}{\partial\eta}\left(\frac{u^2}{2}\right)-\sigma k_6u^2\right]$$

$$\beta_{1\xi}=\frac{2\xi\beta_1}{x\xi'(x)}=2\frac{d\ln u_*}{d\ln\xi},\quad k_6=\frac{\Delta}{\rho H_1\bar{r}^{\nu}R}=\frac{1}{\rho H_1\bar{r}^{\nu}\bar{R}}\sqrt{\frac{2\bar{X}}{K_*}}$$

$$\mathcal{L}=lH_1\bar{r}^{2\nu},\quad l=\frac{\mu\rho}{\mu_*\rho_*},\quad a=\frac{2\xi k_6}{\mathrm{Re}^*\Delta^2x\xi'(x)}=\frac{l\bar{r}^{\nu}}{\rho\bar{R}}\sqrt{\frac{2\bar{X}}{K_*}}$$

$$c=\rho lk_1k_8\frac{x\xi'(x)}{2\xi}=-\frac{x}{(2X)^2}\frac{2\xi}{H_1}\frac{\partial y}{\partial\xi}\frac{v_*}{u_*},\quad \beta_{2\xi}=\frac{2\xi}{\xi'(x)}\frac{d\ln v_*}{d\ln x}=2\frac{d\ln v_*}{d\ln\xi}$$

$$d=\frac{u_*^2\Delta}{H_1\bar{r}^{\nu}V_\infty^2R}=\frac{u_*^2}{H_1\bar{r}^{\nu}V_\infty^2\bar{R}}\sqrt{\frac{2\bar{X}}{K_*}},\quad \bar{R}=R/R_0 \tag{6.14}$$

$$e=\frac{u_*v_*\Delta x\xi'(x)}{xH_1\bar{r}^{\nu}V_\infty^2 2\xi}=\frac{u_*v_*}{H_1\bar{r}^{\nu}V_\infty^2\sqrt{2\bar{X}K_*}},\quad \Delta=R_0\sqrt{\frac{2\bar{X}}{K_*}}$$

The conditions on the shock wave will be (6.8) or (6.9) and (3.20), where Δ is given by the formula presented above. After finding η_s, the standoff distance $y_s=y_s(x)$ is subsequently found from (3.21) using (3.20) if we put $\eta_s=1$ in (3.21).

After solving the VSL equations, the friction coefficient (2.14) and heat transfer coefficient (2.15) will be given by the expressions

$$C_f=\left(\frac{u_*}{V_\infty}\right)^2\sqrt{\frac{2}{\bar{X}K_{**}}}\left(l\frac{\partial u}{\partial\eta}\right)_w$$

$$C_H = \frac{u_*}{V_\infty} \frac{1}{\sqrt{2\bar{X}K_{**}}} \left(\frac{l}{\sigma} \frac{\partial H}{\partial \eta} \right)_w \frac{1}{H_\infty - h_w} \tag{6.15}$$

which agree with formulae (5.21). The overall drag coefficient will be calculated using formula (3.24).

The VSL equations with generalized conditions on the required shock wave describe the flow over the whole of the shock layer both at high and moderate Reynolds numbers.

The advantage of writing the VSL equations in Dorodnitsyn–Lees variables (6.10)–(6.13) is the fact that the influence of second-order effects can be clearly seen from them. In the momenta equation (6.11) these effects are associated with terms containing the coefficients k_6, a and c. Since $k_6 \approx a \sim (\rho_* \mathrm{Re}_*)^{-1/2}$ ($\mathrm{Re}_* = \rho_\infty u_* R_0/\mu_*$), these coefficients are small at high supersonic velocities up to Reynolds numbers $\mathrm{Re}_* \sim 1$. The coefficient $c \sim \rho_*^{-2}$ and the last term in (6.11) can be omitted in the case of high supersonic velocities. The coefficient $\mathcal{L}$ in (6.11) and (6.13) differs from the Rubezin coefficient $l = \mu\rho(\mu_*\rho_*)$ in the boundary-layer equations by the factor $H_1\bar{r}^{2\nu}$, which leads to a weaker dependence of the coefficient $\mathcal{L}$ on η.

If terms in Eq. (6.11) with the coefficients k_6, a and c are omitted, system (6.10)–(6.13) will be only slightly different from the boundary-layer equations. The difference between the problem of the flow around a body, which is solved using a viscous shock layer, and the boundary-layer problem lies in the solution of a similar system of equations but with the determination of the pressure distribution in the shock layer from the additional equations (6.10) and (6.12), which enable one to determine the pressure field over the whole of the shock layer, unlike in the case of boundary-layer theory where the pressure is predetermined from the solution of the problem of the flow of a non-viscous gas around a body.

7. THE EQUATIONS OF A HYPERSONIC (THIN) VISCOUS SHOCK LAYER (TVSL)

This system of equations is obtained from the VSL equations (Section 6) when $k \to 0$ and $\mathrm{Re} \to \infty$ subject to the condition that $k\,\mathrm{Re} = K \sim 1$ [27]. On taking the above-mentioned limit in Eqs (6.10)–(6.13), we obtain

$$\begin{gathered}
u = \frac{\partial f}{\partial \eta}, \quad v_2 = \frac{\xi}{X}\frac{\partial y}{\partial \xi} v_1 - \frac{u_*}{\rho\sqrt{2\bar{X}K_*}}\left(f + 2\xi \frac{\partial f}{\partial \xi} \right) \\
\beta_{1\xi} u^2 + 2\xi u \frac{\partial u}{\partial \xi} - \left(f + 2\xi \frac{\partial f}{\partial \xi}\right)\frac{\partial u}{\partial \eta} = -\frac{V_\infty^2 2\xi}{\rho u_*^2}\frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \eta}\left(l \frac{\partial u}{\partial \eta} \right) \\
\frac{\partial p}{\partial \eta} = \frac{u_*^2 u^2}{V_\infty^2 R}\sqrt{\frac{2\bar{X}}{K_{**}}} = \frac{u_* u^2 \sqrt{2\xi}}{\rho_\infty V_\infty^2 R r_w^\nu}, \quad K_{**} = \frac{\rho_\infty u_* R_0}{\mu_* \rho_*} \\
2\xi u \frac{\partial H}{\partial \xi} - \left(f + 2\xi\frac{\partial f}{\partial \xi}\right)\frac{\partial H}{\partial \eta} = \frac{\partial}{\partial \eta}\left(l \frac{\partial H}{\partial \eta}\right)
\end{gathered} \tag{7.1}$$

The boundary conditions on the body for this system will be (3.16). The boundary conditions in the shock wave (6.8) take the simpler form

$$\begin{gathered}
u(\xi, \eta_s) = 1 - \left(\frac{1}{K\bar{\Delta}\sin\beta}\frac{\partial u}{\partial \eta}\right)_s, \quad p(\xi, \eta_s) = \sin^2\beta \\
H(\xi, \eta_s) = H_\infty - \frac{1}{K\bar{\Delta}\sin\beta}\left[\frac{1}{\sigma}\frac{\partial H}{\partial \eta} + u_*^2\left(1 - \frac{1}{\sigma}\right)\frac{\partial}{\partial \eta}\left(\frac{u^2}{2}\right)\right] \\
\bar{\Delta} = \sqrt{\frac{2\bar{X}}{K_*}}, \quad u_i = u_* = V_\infty \cos\beta
\end{gathered} \tag{7.2}$$

Here, in the conditions for u and H, the angle β is not replaced by α, breaking the sequence of the asymptotic approach. Henceforth, we may always put $\beta = \alpha$. The unknown "standoff distance" of the shock wave η_s will be found from condition (3.20) which, in the approximation being considered, will be

$$f(\xi,\eta_s) = \frac{r_w}{(\nu+1)R_0\cos\beta}\sqrt{\frac{K_*}{2\bar{X}}} = \frac{\rho_\infty V_\infty r_w^{\nu+1}}{(\nu+1)\sqrt{2\xi}} \tag{7.3}$$

In model (7.1)–(7.3), the effects of the second approximation of boundary-layer theory associated with the longitudinal and transverse curvature drop out. However, the vortex-interaction effects remain [27] since the solution is constructed at once over the whole domain from the body to the shock wave.

The velocity v_2, the coordinate $y = y(\xi, \eta)$ and the standoff distance $y_s = y_s(x)$ can be determined after problem (7.1)–(7.3) has been solved. Problem (7.1)–(7.3) is solved by a marching method, beginning from the stagnation line $\xi = 0$. Many specific problems both for a perfect gas as well as when allowance is made for the real physicochemical processes occurring in the shock layer have been solved within the framework of this model [5]. The advantages and disadvantages of this model have been discussed in [5, 15].

8. THE PARABOLIZED NAVIER–STOKES EQUATIONS

In the case of moderate and low Reynolds numbers, rather than using a two-layer scheme (a shock layer plus the structure of the shock wave) in the numerical computation of supersonic flow around a body, it is more convenient to obtain the solution at once over the whole of the perturbed flow domain without separating the shock wave. In such a direct calculation the Navier–Stokes structure is obtained when solving the problem [22]. Such a direct solution can be obtained numerically at comparatively low Reynolds numbers when the shock-wave thickness, which is proportional to Re^{-1}, becomes comparable with the whole of the perturbed flow domain and there is no need for any additional subdivision of the mesh in the shock wave structure. In this case, the four conditions in the free stream (3.17) and the three conditions on the wall (3.16) are sufficient to solve the problem of supersonic flow around a body using the Navier–Stokes equations. The VSL equations (see Section 6) are one order lower with respect to the coordinate η in Eq. (3.1) which are of the order of $O(\mathrm{Re}^{-1})$. Here, the second derivative of u with respect to η are omitted since, on passing through the shock wave, the derivative $\partial u/\partial\eta$ does not undergo a sharp change, although the terms with the second derivatives are formally of the same order as the terms with the derivatives $\partial^2 v/\partial\eta^2$. Then, instead of (3.10), we obtain the simplified equation for the y-axis projection of the momenta.

$$\beta_2 uv - \frac{1}{k_4}u^2 + x\xi'(x)u\frac{\partial v}{\partial\xi} - \left(\beta_0 f + x\xi'(x)\frac{\partial f}{\partial\xi}\right)\frac{\partial v}{\partial\eta} = \\ = -\frac{1}{k_5}\frac{\partial p}{\partial\eta} + \frac{\partial}{\partial\eta}\left[\varepsilon_3\left(\zeta+\frac{4}{3}\right)\rho k_2\frac{\partial v}{\partial\eta}\right] \tag{8.1}$$

On changing in this equation to the variables (5.12), we obtain

$$\beta_{2\xi} uv - \frac{1}{k_{4\xi}}u^2 + 2\xi u\frac{\partial v}{\partial\xi} - \left(f + 2\xi\frac{\partial f}{\partial\xi}\right)\frac{\partial v}{\partial\eta} = \\ = -\frac{1}{k_{5\xi}}\frac{\partial p}{\partial\eta} + \frac{\partial}{\partial\eta}\left[\left(\zeta+\frac{4}{3}\right)\mathcal{L}\frac{\partial v}{\partial\eta}\right] \tag{8.2}$$

where

$$k_{4\xi} = \frac{Rv_*}{2Xu_*} = -\frac{R\,\mathrm{tg}\,\alpha}{2X}, \quad k_{5\xi} = \frac{u_*\nu_*\Delta}{2XH_1\bar{r}^\nu V_\infty^2} = -\frac{\Delta\sin\alpha\cos\alpha}{2XH_1\bar{r}^\nu}$$

Here, we have put: u_* and $v_{1\infty} = V_\infty \cos\alpha$, $v_* = v_{2\infty} = -V_\infty \sin\alpha$. Equations (6.10), (6.11) and (6.13) will be the remaining equations of the parabolized system of Navier–Stokes equations. This system, like the VSL equation, retains elliptic properties in subsonic flow domains [22].

9. A VANISHINGLY THIN VISCOUS SHOCK LAYER (Re $\to$ 0, $k \to 0$, SUBJECT TO THE CONDITION THAT $(k/\mathrm{Re})^{1/2} \to 0$)

Unlike all the preceding models in which Re $\to \infty$, we shall assume in this model that Re $\to$ 0 and that the shock-layer thickness is vanishingly small ($k \to 0$).

From the very outset of the arousal of interest in the problem of the hypersonic flow of a rarefied gas around bodies (in the 1950s) the idea that the continuum (hydrodynamic) approach was invalid in the aerodynamics of transitional and free molecular flow conditions around a body, that is, when the Knudsen number Kn is of the order of unity and greater, became the widely held and prevalent point of view among specialists in gas dynamics and molecular dynamics. This was due to at least two circumstances.

Firstly, the asymptotic derivation of the equations of hydrodynamics from the kinetic Boltzmann equation using the Chapman–Enskog method indicated that the Navier–Stokes equations hold up to terms $O(\mathrm{Kn})$ inclusive. However, it should be noted that taking account of the terms $O(\mathrm{Kn})$ in the Navier–Stokes equations is not of an asymptotic character but of a composite character. The fact that they agree closely with the results based on the linear irreversible thermodynamics, their applicability in efficient direct calculations of the whole of the flow field with a shock wave and the frequently unexpected wider range of applicability with respect to Kn numbers than would follow from their asymptotic derivation, is the justification for such an "inconsistent" derivation. The description of rarefied gas flows using continuum models is preferable since such models require much less computational resources than models based on a molecular approach. Furthermore, when account is taken of physicochemical processes in the shock layer, the continuum approach does not require any knowledge of the rate constants for the elementary acts of interaction between the particles and, in the continuum approach, a knowledge of the averaged macroscopic rate constants is sufficient.

Secondly, the Navier–Stokes structure of the shock wave at high Mach numbers differs from the structure obtained from the solution of kinetic equations and from experiment.

Repeated attempts have therefore been made to take account of higher terms in the Chapman–Enskog expansion of the distribution function with respect to Kn numbers, which leads to continuum models in the form of the Burnett equations and super-Burnett equations, which contain higher derivatives of the hydrodynamic variables. Thus when solving the boundary-value problems with these equations there is a difficulty of formulating the additional boundary conditions (which do not arise from the mechanical formulation of the problem) as well as the occurrence of instability in the numerical solutions when the mesh step size is subdivided. Furthermore, numerical solution of the problem of the flow around a plate with a sharp leading edge has shown [32] that the Burnett equations provide a less accurate description of the flow field than the Navier–Stokes equations. This enables one to hope that the Burnett equations can only improve the solution when the Navier–Stokes equations are of high accuracy [18]. Hence, at the present time, there are no continuum models capable of describing flows with high Knudsen numbers in problems of hypersonic flow around bodies.

However, what has been said above refers to the description of flow domains with Kn numbers of the order of unity or greater. Here, the possibility of using continuum approaches to the description of flow domains, which are vanishingly small in size with small Kn numbers, where the continuum approach remains valid, has not been ruled out. Such flows must be characterized by low Re numbers. We will therefore next make an attempt in this paper to consider the problem of hypersonic flow around bodies when Re $\to$ 0 and $k \to 0$ (VTVSL) within the framework of the Navier–Stokes equations. The condition

$$\sqrt{k/\mathrm{Re}} \to 0 \tag{9.1}$$

has to be imposed in order to obtain physically correct results.

In particular, since flows with large coefficients of viscosity μ correspond to low Re numbers, this means that the terms containing the viscosity in the Navier–Stokes equations (3.6), (3.9)–(3.11) are the principal ones. We therefore estimate all the terms in the Navier–Stokes equations with respect to the first term containing the viscosity in Eq. (3.9), taking $\mathrm{Re}^*\Delta^2 \sim 1$, whence $\bar{\Delta} \sim (K)^{-1/2}\bar{x}H_1r^{-2\nu}$ ($\bar{x} = x/R_0$). Using estimate (4.2), which holds everywhere, we find $y \sim R_0(k/\mathrm{Re})^{1/2}$. From the necessary condition

(9.1), we then obtain $H_1, \bar{r} \sim 1$. Henceforth, we shall therefore use the estimate

$$\bar{\Delta} \sim K^{-1/2} \tag{9.2}$$

It is interesting to note that, in appearance, (9.2) is similar to estimate (5.3) obtained for a boundary layer. However, these estimates are basically different from one another. Whereas in (5.3), $K = k\,\mathrm{Re} \to \infty$, in (9.2) $K \to 0$. From the condition on the separating line (3.20) or from the second equation of (3.6), we obtain

$$f \sim u \sim \bar{\Delta}^{-1} \sim K^{1/2}, \quad v \sim 1 \tag{9.3}$$

that is, the tangential component of the velocity in a VTVSL tends to zero when $\mathrm{Re} \to 0$ and $k \to 0$ as $K^{1/2}$. This means that the momentum along the x axis does not change in a shock layer of vanishing thickness. Next, on taking account of the fact that we have $\upsilon_i \sim k$ on the separating line $y_s = y_s(x)$ (see (2.9) and (2.10)) and taking $\upsilon_* = \upsilon_i \sim k(u_* \sim V_\infty)$ as the characteristic velocity, we obtain the following estimates for the coefficients k_i ($i = 1, \ldots 8$) and ε_i ($i = 1, \ldots 5$)

$$k_1 \sim k_3 \sim k_4 \sim k_7 \sim k, \quad k_2 \sim kK^{1/2} \tag{9.4}$$

$$k_5 \sim k_6 \sim kK^{-1/2} \sim (k/\mathrm{Re})^{1/2}, \quad k_8 \sim k(k/\mathrm{Re})^{1/2}$$

$$\varepsilon_1 \sim \varepsilon_2 \sim k/\mathrm{Re}, \quad \varepsilon_3 \sim K^{-1/2}, \quad \varepsilon_4 \sim \varepsilon_5 \sim \mathrm{Re}^{-1} \tag{9.5}$$

The components τ_{ij} ($i, j = \xi, \eta, \zeta$) and the fluxes X, X', Y from (3.12) will be of the following orders

$$\tau_{\xi\xi} \sim \tau_{\eta\eta} \sim \tau_{\zeta\zeta} \sim \tau_{\xi\eta} \sim \tau'_{\xi\xi} \sim \tau'_{\zeta\eta} \sim \sqrt{K} \tag{9.6}$$

Using condition (9.1) and estimates (9.2)–(9.6), the system of Navier–Stokes equations (3.6), (3.9)–(3.11) takes the simplest possible form

$$u = \frac{\partial f}{\partial \eta}, \quad \rho v_2 = -\frac{\Delta u_*}{2X}\left(f + 2\xi\frac{\partial f}{\partial \xi}\right) \tag{9.7}$$

$$\frac{\partial}{\partial \eta}\left(l\frac{\partial u}{\partial \eta}\right) = 0, \quad l = \frac{\mu\rho}{\mu_*\rho_*} \tag{9.8}$$

$$\frac{\partial p}{\partial \eta} = 0 \tag{9.9}$$

$$\frac{\partial}{\partial \eta}\left(\frac{l}{\sigma}\frac{\partial H}{\partial \eta}\right) = 0 \tag{9.10}$$

The boundary conditions on the wall remain in the previous form (3.16). The boundary conditions on the separating line $y_s = y_s(x)$ (3.18), (3.29) are also very much simplified ($u_* = u_i = V_\infty \cos\alpha$) and take the form

$$\frac{1}{K_*\bar{\Delta}\sin\beta}\left(l\frac{\partial u}{\partial \eta}\right)_s = 1, \quad p_s = \sin^2\beta$$

$$\frac{1}{K_*\bar{\Delta}\sin\beta}\left(\frac{l}{\sigma}\frac{\partial u}{\partial \eta}\right)_s = H_\infty, \quad f_s = \frac{r_w}{(\nu+1)\Delta\cos\beta} \tag{9.11}$$

The second equation of (9.7) is self-contained and, after solving problem (9.7)–(9.10), (3.16), (9.11), the velocity $\upsilon_2 \sim (k/\mathrm{Re})^{1/2}$ is determined from it.

For convenience in obtaining the solution, we normalize the shock-layer thickness by putting

$$\Delta = \int_0^{y_s(x)} \rho\, dy$$

Then, $0 \leq \eta \leq 1$. The solution of the problem is written out in the quadratures

$$\Delta^2 = \frac{r_w R_0}{(\nu+1)K_* I \sin\beta\cos\beta}, \quad I = \int_0^1 \Phi_1(\eta)d\eta, \quad \Phi_1(\eta) = \int_0^\eta \frac{d\eta'}{l}$$

$$u = K_* \bar{\Delta}\Phi_1(\eta)\sin\beta, \quad p = \sin^2\beta \tag{9.12}$$

$$\frac{H - h_w}{H_\infty} = \frac{h - h_w}{H_\infty} = K_* \bar{\Delta}\Phi_2(\eta)\sin\beta, \quad \Phi_2(\eta) = \int_0^\eta \frac{\sigma d\eta'}{l}$$

The pressure coefficient, the friction coefficient and the heat transfer coefficient will be

$$C_p = 2\sin^2\beta, \quad C_f = 2\sin\beta\cos\beta, \quad C_H = \frac{\sin\beta}{1 - h_w / H_\infty} \tag{9.13}$$

Under the assumption that $l \approx 1$, we obtain

$$y_s = \Delta(x)\int_0^1 \frac{d\eta}{\rho} = \frac{(\gamma-1)R_0}{2\gamma p}\left[\bar{\Delta}\frac{h_w}{H_\infty} + \frac{\sigma \bar{r}_w}{(\nu+1)\cos\beta}\right], \quad \bar{r}_w = r_w / R_0$$

The coefficients (9.13) agree exactly with the corresponding coefficients in the case of free molecular flow around a body with accommodation coefficients equal to unity. The expression for C_p is Newton's law, which does not follow from Euler's equations when $k \to 0$. It is well known that, when $k \to 0$, the Busemann formula, which takes account of the effect of centrifugal forces in the shock layer, is obtained instead of it. It follows from what has been described above that, when viscosity is taken into account, these forces are proportional to $K/k = \mathrm{Re}$ and tend to zero as $\mathrm{Re} \to 0$. Hence, Newton's law of resistance follows from the Navier–Stokes equations when $k \to 0$, $\mathrm{Re} \to 0$ subject to the condition that $(k/\mathrm{Re})^{1/2} \to 0$ or, in other words, from VTVSL theory.

It follows from the solution (9.12) that, when condition (9.1) is satisfied, subject to which it was obtained, we have a vanishingly thin viscous shock layer (VTVSL) (when $h_w/H_\infty \neq 0$), the thickness of which tends to zero as

$$y_s \sim k\Delta \sim \frac{k}{\sqrt{K}} = \sqrt{\frac{k}{\mathrm{Re}}} \to 0 \tag{9.14}$$

If the Kn number is defined as the ratio of the mean free path to the shock-layer thickness y_s we obtain $\mathrm{Kn} = l/y_s \sim K^{1/2} \to 0$. Consequently, in the solution obtained, the gas motion after shock wave passes occurs in an extremely thin shock layer ($y_s \to 0$), in which the x axis projection of the momentum of the free stream and the enthalpy do not change over the thickness of the shock layer and are transferred with the shock wave onto the body as a whole (see (9.8), (9.10), (9.11)).

Graphs of C_H in the neighbourhood of the critical point, obtained by the numerical solution of the VTVSL, VSL and Navier–Stokes (NS) equations, are shown in Fig. 2 as a function of $\mathrm{Re} = (\rho_\infty V_\infty R_0)/\mu(T_0)$. In the VTVSL equations, condition (9.1) is satisfied and $C_H \to 1$ when $\mathrm{Re} \to 0$. In the VSL equations, condition (9.1) is not satisfied, starting from a certain small Re number (when $\mathrm{Re} \approx 10$ in the given case) and $C_H \sim \mathrm{Re}^{-1/2} \to \infty$, which confirms the above theory.

It should be noted that the fact that the flow reaches free molecular behaviour has been noted previously [27, 33], starting out from a particular exact solution of the thin viscous shock layer (TVSL) equations, which hold in the neighbourhood of the critical point of an axially symmetric body when the viscosity depends linearly on the temperature. The results obtained here are of a general nature, they do not depend on the actual transport properties of the gas and they hold for both axially symmetric and plane two-dimensional problems. It has been shown that, when $k \to 0$, $\mathrm{Re} \to 0$, subject to the condition that $(k/\mathrm{Re})^{1/2} \to 0$, the problem of hypersonic flow around a body reduces to the solution of the simplified ("local") Stokes equations (9.7)–(9.10) in a vanishingly thin viscous layer with the required boundary (condition (9.12)).

Hence, the problem of supersonic and hypersonic flow around a blunt body along the whole of a trajectory for entry into the Earth's atmosphere (or into the atmosphere of a planet), from free molecular flow conditions to continuum flow conditions, corresponding to high Re numbers close to the surface of a planet, can be solved using a VSL model (Section 6), by means of a single numerical algorithm

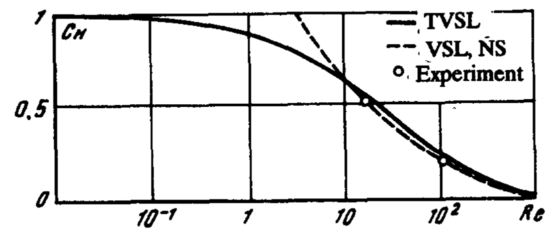

Fig. 2.

such as those in [11] or [12]. In this case, at low Re numbers, it is necessary to observe the not very burdensome condition (9.1) artificially. Actually, since, at high altitudes, the entry velocities of space vehicles and cosmic bodies are high (of the order of 10 km/s), we have $k \sim 0.05$. It is therefore necessary to use condition (9.1) at extremely low Re numbers which, for bodies with $R_0 \sim 1$ m, corresponds to altitudes $H \sim 100$–150 km above the Earth's surface. At low Re numbers, the structure of the shock wave (the second layer in the two-layer model) must be found separately from the solution of the kinetic equations or from the corresponding macroscopic equations.

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